ESRI Discussion Paper Series No.224

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An application to the Japanese economy

by

Stéphane Adjemian
and
Michel Juillard

October 2009

Economic and Social Research Institute
Cabinet Office
Tokyo, Japan

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Stéphane Adjemian and Michel Juillard,CEPREMAP

Abstract

In this paper we present a methodology to deal with trends in DSGE models. This type of models have consequences for long run growth as well as for cyclical dynamics and it would be desirable to deal with both aspects in an unified frame-work. Two different problems need to be addressed. The first ones concerns a rigorous local approximation of a balanced growth model. This is solved by the usual practice of stationarizing the model first, and using trends directly in(log-)linear form.

The second issues deals with the estimation in the level of the data. When the data are not stationary, it is necessary to use a diffuse Kalman filter as the one proposed for example by Koopman and Durbin (2003).In this paper, we propose a modification of this filter in order to better deal with cointegrated variables.

As an illustration, we develop a medium size New Keynesian model with consumption habits, adjustment costs and nominal rigidities in the goods and labor markets. This model is estimated on Japanese data.

In order to take into account the zero nominal interest rate bound to which the Japanese monetary policy was confronted in the recent period, we experiment with modeling the log of the interest rate rather than the interest rate itself. This workaround is far from perfect, but it permits the use of local approximation without having a model generating negative values for the nominal interest rate.

Dealing with trends in DSGE models: An application to the Japanese economy

Stéphane Adjemian and Michel Juillard*†

March 2009

1 Introduction

Depending on the questions that they ask, economists focus more on determinants of long run growth or, on the contrary, on t he mechanisms of short run dynamics. However, it is obvious that a given model has usually implications both for fluctuations at business cycle frequencies and for the long run. This is even truer in Dynamic Stochastic General Equilibrium models (DSGE) that have explicit microeconomic foundations. Long term growth is generally induced by technical progress and demographic change.

Traditionally, the issue has been addressed in two ways, either the business cycle model is written without reference to the growth factors, but, then, the data to which the model is confronted must be detrended by mean of a statistical method. Or, the original data are used and the model contains a growth mechanism. It can be argued that the first approach is less satisfactory, because the growth component that is removed by the statistical detrending procedure isn't consistent with the growth mechanism implicitly implied by the theoretical model. However, estimating DSGE models with growth trends raises additional methodological problems that must be carefully addressed.

The first problem concerns the solution method used to solve the DSGE models. While it isn't the only option, in practice, the solution involves a local approximation of the original, structural, model. Obviously, it wouldn't be very reasonable to compute the local approximation of

*CEPREMAP, 142 rue Chevaleret, 75013 Paris. **e-mail:** michel [DOT] juillard [AT] ens [DOT] fr

†We thank Antoine Devulder for many discussions and having shared his modeling ideas.

a model with growing variables around a given point: the quality of the local approximation would strongly deteriorate as one gets further from the point around which the local approximation is computed! The solution to this problem is to compute the local approximation around a stationarized version of the model. Such a version exists if the model is characterized by balanced growth.

In order to proceed with estimation, it is necessary to express the bridge between stationarized and original variables. It is then possible to estimate the model against original variables. Again, a choice opens: one can choose to estimates in growth rates or in levels of the original variables. The latter provides additional information on the co-integrating relationships of the model.

Finally, when growth dynamics take the form of non-stationary processes and one wants to estimate against data in levels of the variables, an additional issue arises with the initialization of the Kalman filter that is used to compute the likelihood of the model. A frequent recommendation is to use a diffuse filter.

As an illustration of this procedure, we present then a medium size New Keynesian model, inspired from **?**[1]. There is an explicit mechanism for exogenous growth of labor embodied technological change.

2 Dealing with trends in DSGE models

In a first sub–section, we discuss the process through which one can stationarized a balanced growth model and take–up in section 2 problems related to the use of the diffuse Kalman filter.

2.1 A stationary transformation

As already mentioned, there exists a specific problem with nonlinear growth models. Current estimation methods require a local (log–)linear approximation of the original model. In the case of stationary models, it is reasonable to take the local approximation around the deterministic steady state of the model. The deterministic steady state is the point of the state space where the model would converge in the absence of shocks. This method is legitimate as long as the shocks

[1]A similar type of model has also been estimated on Japanese data in **?**

aren't too large.

However, by definition, there is no steady state point for a growth model. Furthermore, if one was to take a local approximation of a growth model around any particular point, it is obvious that the quality of this local approximation would deteriorate as the economy moves further away of this particular point. This would be true as time goes by, but also as one would move further towards the past.

A similar preoccupation is in order in a framework wider than economic growth proper, when some variables of the model follow a non-stationary process. Even without drift, a unit root process can move very far from a given point. For example, most models with an inflation target rule will induce a stochastic trend in the price level, even if the inflation target is zero. When the inflation target is greater than zero, the price level will be characterized with both a pure stochastic trend and a deterministic one.

Intuitively, if a model possess a reference growth path, one would want to compute a local approximation around this path. This is what is done when one stationarize a model characterized by a balanced growth path. A model is characterized by balanced growth when, in the absence of shocks, each variable converges towards a path with a constant rate of growth. Of course, some variables can have a constant long run path with a null growth rate.

In this class of models, it is by definition always possible to define the relative distance to the trend:

$$\widehat{Y}_t = \frac{Y_t}{\bar{Y}_t}$$

$$\bar{Y}_t = (1 + g_{Y\,t})\,\bar{Y}_{t-1}$$

where Y_t is the original variable, $\bar{Y}_t$, the value of the trend at the same period, $\widehat{Y}_t$, the relative distance to the trend or *stationarized* variable, and $g_{Y\,t}$ is the growth rate of the trend.

It is then possible to rewrite the model in term of the stationarized variables. When replacing the original variables by stationarized ones, the following rules must be followed, depending on the relative time period at which the variable appears in the model:

1. variables appearing at the current period, Y_t, are replaced by their stationarized counterpart, $\widehat{Y}_t$.

2. variables appearing with a lag, Y_{t-k}, are replaced by their stationarized counterpart, divided by their growth factor elevated to the power of the number of lags, $\frac{\widehat{Y}_t}{(1+g_Y)^k}$.

3. variables appearing with a lead, Y_{t+k}, are replaced by their stationarized counterpart, multiplied by their growth factor elevated to the power of the number of leads, $(1+g_Y)^k \widehat{Y}_t$.

Another feature of balanced growth models that is useful in this context is the following. If one replaces the original variables with its expression as the product of the stationarized variable and its trend, $Y_t = \widehat{Y}_t \bar{Y}_t$, it is then possible to eliminate the trend variables. In practice, it is a useful property that lets one verify that the model has indeed a balanced growth path.

When one estimates a model with non-stationary variables, a further problem arises with the initialization of the Kalman filter that is used in order to compute the likelihood of the model.

2.2 The diffuse filter

The estimation strategy is as follows. The first order approximated solution of the stationarized model described above provides an equation of the form

$$\hat{y}_t = A\hat{y}^s_{t-1} + Bu_t$$

where $\hat{y}_t = y_t - \bar{y}$ is the vector of variables of the model in deviation to the steady state, y^s_t are the variables present in the state vector (all the variables appearing with a lag in the model), u_t are the shocks, and A and B are the coefficients of this reduced form.

Because only some of the variables present in the model are observed, we are dealing here with a statistical model of unobserved components. Its likelihood is computed by the Kalman filter and the model must be put in state space form.

The transition equation describes the dynamics of the state variables:

$$\hat{y}^{(1)}_t = A^{(1)}\hat{y}^{(1)}_{t-1} + B^{(1)}u_t$$

where $A^{(1)}$ and $B^{(1)}$ are the appropriate sub-matrices of A and B, respectively. $y_t^{(1)}$ is the union of the state variables y_t^s, including all necessary lags, and $y_t^\star$, the observed variables. Note that matrix $A^{(1)}$ can have eigenvalues equal to one.

The measurement equation is written

$$y_t^\star = \bar{y} + M\hat{y}_t^{(1)} + x_t + \epsilon_t$$

where M is the selection matrix that recovers $\hat{y}_t^\star$ out of $\hat{y}_t^{(1)}$, x_t is a deterministic component[2] and ϵ_t is a vector of measurement errors.

In addition, we have, the two following covariance matrices, for the structural shocks and for the measurement errors:

$$\begin{aligned} E\left(u_t u_t'\right) &= Q \\ E\left(\epsilon_t \epsilon_t'\right) &= H \end{aligned}$$

In what follows, it is simpler and computationally more efficient to remove directly the deterministic elements from the observations before applying the filter. The simplified measurement equation is then

$$\tilde{y}_t^\star = M\hat{y}_t^{(1)} + \epsilon_t$$

with $\tilde{y}_t^\star = y_t^\star - \bar{y} - x_t$.

When the stochastic trends in the model contain a deterministic component, it is necessary to separate the pure random walk process that is represented in the transition equation from its deterministic component that is treated as an exogenous element in the measurement equation.

The initialization of the Kalman filter remains an important issue. For stationary models, the state variables are initialized at their unconditional mean and the variance of the one–step ahead forecast errors in first period are set equal to the unconditional variance of the variables. This option isn't available in non-stationary models as the unconditional variance is infinite.

Durbin and Koopman (2004) propose a diffuse filter to handle this situation. The basic idea is

[2] Currently, Dynare only accommodates linear trends.

to compute the limit of the Kalman filter when initial variance tends toward infinity. Essentially, they analyze the filter when P_0, the initial variance of the one–step ahead forecast error is the following limit:

$$P_0 = \lim_{k\to\infty} kI$$

This approach is however problematic in the case of co-integration relationships, because the above limit for the variance matrix doesn't take into account the relations between co-integrated variables.

For this reason, we propose to apply the diffuse filter to a transformed state space representation where the independent stochastic trends appear directly. This transformed system is obtained by a Schur decomposition of the transition matrix.

In the transition equation

$$\hat{y}_t^{(1)} = A^{(1)}\hat{y}_{t-1}^{(1)} + B^{(1)}u_t$$

we propose to perform a reordered real Schur decomposition on transition matrix $A^{(1)}$:

$$A^{(1)} = W \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} W'$$

where T_{11} and T_{22} and quasi upper–triangular matrices and W is an orthogonal matrix. The reordering is such that the absolute value of the eigenvalues of T_{11} are all equal to 1 while the eigenvalues of T_{22} are all smaller than 1 in modulus. When there are co-integrating relationships between the state variables, there are obviously less unit roots in the system than the number of non-stationary variables in the model. The dimension of T_{11} reflects this fact.

It is then natural to rewrite the transition equation in transformed variables as

$$a_t = Ta_{t-1} + Ru_t$$

where $a_t = W'\hat{y}_t^{(1)}$ and $R = W'B$. The measurement equation becomes

$$\tilde{y}_t^\star = Za_t + \epsilon_t$$

with $Z = MW$. Note that in this formulation of the state space representation, only the state variables are transformed, structural shocks and measurement errors stay the same as in the original formulation.

The diffuse initialization of the filter is now as follows. The initial values for the state variables are $a_0 = 0$. This is the unconditional mean of the stationary elements in a_t and has no effects for the non-stationary ones.

Following Durbin and Koopman, we set

$$\begin{aligned} P_0 &= P_0^{\infty} + P_0^{\star} \\ &= \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \Sigma_{\tilde{a}} \end{bmatrix} \end{aligned}$$

where I is an identity matrix of the same dimensions as T_{11}. It corresponds to the diffuse prior on the initial values of the stochastic trends. $\Sigma_{\tilde{a}}$ is the covariance matrix of the stationary part of a_t. $\Sigma_{\tilde{a}}$ is the covariance matrix of $\tilde{a}_t$ with dynamics

$$\tilde{a}_t = T_{22}\tilde{a}_{t-1} + \tilde{R}\eta_t$$

or

$$\Sigma_{\tilde{a}} = T_{22}\Sigma_{\tilde{a}}T_{22}' + \tilde{R}Q\tilde{R}'$$

where $\tilde{R}$ is the conforming sub-matrix of R. As T_{22} is already quasi upper–triangular, it is only necessary to use part of the usual algorithm for the Lyapunov equation.

While P_t^{∞} is different from zero, the filter (and smoother) is in a diffuse step. When $t > d$, where d is the maximum integration order, the procedure falls back on standard recursions.

At $t = 0$

$$E\left(a_{1|0}\right) = P_{1|0} = P_{1|0}^{\infty} + P_{1|0}^{\star}$$

Then,

$$
\begin{aligned}
F_t^{\infty} &= Z P_{t|t-1}^{\infty} Z' \\
F_t^{\star} &= Z P_t^{\star} Z' + H \\
K_t^{\infty} &= T P_{t|t-1}^{\infty} Z' \left(F_t^{\infty}\right)^{-1} \\
K_t^{\star} &= T \left(P_{t|t-1}^{\star} Z' \left(F_t^{\infty}\right)^{-1} - P_{t|t-1}^{\infty} Z' \left(F_t^{\infty}\right)^{-1} F_t^{\star} \left(F_t^{\infty}\right)^{-1} \right) \\
v_t &= \tilde{y}_t^{\star} - Z a_{t|t-1} \\
a_{t+1|t} &= T a_{t|t-1} + K_t^{\infty} v_t \\
P_{t+1|t}^{\infty} &= T P_{t|t-1}^{\infty} \left(T' - Z' K_t^{\infty\prime}\right) \\
P_{t+1|t}^{\star} &= -T P_{t|t-1}^{\infty} Z' K_t^{\star\prime} {}^{\star}_{t|t-1} \left(T' - Z' K_t^{\infty\prime}\right) + RQR'
\end{aligned}
$$

where $a_{t|t-1} = E_{t-1} a_t$.

Finally, the log–likelihood is given by

$$
-\frac{nT}{2} \ln 2\pi - \frac{1}{2} \sum_{t=1}^{T} \ln |F_t| - \frac{1}{2} \sum_{t=1}^{T} v_t' F_t^{-1} v_t
$$

3 A DSGE model

What follows describes a DSGE model whose structure is close to **?**.

3.1 Households

The economy is populated with a continuum of households $h \in [0,1]$. Each household values consumption of a composite good. We write $C_t(h)$ the demande of this good by household h in period t. A household offers as well labor hours. We write $L_t(h)$ the labor supply of household h in period t. Welfare is defined as:

$$
\begin{aligned}
&\mathcal{W}_t(h) = u_t(h) + \beta \mathbb{E}[\mathcal{W}_{t+1}(h)] \\
&u_t(h) = \frac{\left[C_t(h) - \eta \overline{C}_{t-1}\right]^{1-\sigma_c}}{1-\sigma_c} \exp\left\{ \varepsilon_{L,t} \frac{\sigma_c - 1}{1+\sigma_l} L_t(h)^{1+\sigma_l} \right\}
\end{aligned}
\tag{3.1}
$$

where $\varepsilon_{L,t}$ is a shock to labor supply. $\log \varepsilon_{L,t}$ is an AR exogenous shock process with mean $\log \tilde{L}$ (this parameter gives us an extra degree of freedom for adjusting the stationary level of hours). We choose this form for the utility function in order to build a model compatible with balanced growth.

We assume that utility obtained by household h in period t depends not only on its own consumption but as well on aggregate consumption in previous period, $\overline{C}_{t-1} = \int_0^1 C_{t-1}(h)\mathrm{d}h$. This is a mechanism of external habits.

The budget constraint of household h, in period t, in real terms, is the following:

$$\begin{aligned} C_t(h) + p_{I,t} I_t(h) = &\left\{ \frac{B_{t-1}(h)}{P_t} - \frac{B_t(h)}{P_t \varepsilon_{B,t} R_t} + (1-\tau_{W,t}) \frac{W_t^m}{P_t} L_t(h) \right. \\ &\left. + r_t^k z_t(h) K_{t-1}(h) + \frac{\mathscr{D}_{1,t}(h) + \mathscr{D}_{2,t}(h)}{P_t} \right\} + T_t \end{aligned} \tag{3.2}$$

where P_t is the aggregate price index; $R_t = 1 + i_t$, corresponds to the rate of interest plus one, $B_t(h)$ the nominal value of bonds detained by household h at the end of period t, ε_t^B is the risk premium requested by households in order to detain the bond; $\log \varepsilon_t^B$ is an AR process with zero mean; $I_t(h)$ is investment of h during period t; $\log p_{I,t}$ is an exogenous shock on the relative price of investment and follows an AR process with zero mean; W_t^m is the hourly wage rate received by household h in period t; T_t represents net transfers received by the household during the period; $\mathscr{D}_{1,t}(h)$ and $\mathscr{D}_{2,t}(h)$ are the dividends received from firms and from the *unions* that differentiate household labor supply.

On the resource side, return on physical capital, r_t, is given by

$$r_t^k = z_t(h) K_{t-1}(h)$$

where the stock of physical capital at date t is

$$K_t(h) = (1 - \delta(z_t(h))) K_{t-1}(h) + \varepsilon_{I,t} \left(1 - \mathcal{S}\left(\frac{I_t(h)}{I_{t-1}(h)} \right) \right) I_t(h) \tag{3.3}$$

$z_t(h) \in [0,1]$ is the rate of utilization of physical capital with steady value $z^\star$; the depreciation rate, δ, is a function of the rate of utilization that verifies $\delta(0) = 0$, $\delta(1) = 1$, $\delta(z)' > 0$ for all $z \in [0,1]$ and we write $\delta(z^\star) = \delta^\star$; $\varepsilon_{I,t}$ is a random shock to the efficiency of capital accumulation $\log \varepsilon_{I,t}$ is an AR process with zero mean; function $\mathcal{S}$ describes adjustment costs on investment, we assume $\mathcal{S}(1+g) = 0$, where g is the rate of growth of investment on the balanced growth path, furthermore $\mathcal{S}(1+g)' = 0$ and $\mathcal{S}'' > 0$.

Each household h chooses its consumption, labor supply, bond holdings, investment, and capital utilization rate so as to maximize its inter-temporal utility (3.1) under the budget constraint (3.2) and the the law of evolution of physical captital (3.3), taking as given evolution of prices and exogenous variables.

The first order optimality conditions are given by:

$$\left(C_t(h) - \eta \overline{C}_{t-1}\right)^{-\sigma_c} \exp\left\{\varepsilon_{L,t} \frac{\sigma_c - 1}{1+\sigma_l} L_t(h)^{1+\sigma_l}\right\} = \lambda_t(h) \tag{3.4}$$

where $\lambda_t(h)$ is the Lagrange multiplier associated to the real budget constraint,

$$\lambda_t(h) = \beta \varepsilon_{B,t} R_t \mathbb{E}_t \left[\frac{\lambda_{t+1}(h)}{\pi_{t+1}}\right] \tag{3.5}$$

where $\pi_{t+1} \equiv P_{t+1}/P_t$ is the inflation rate between period t and $t+1$,

$$u_t(h)\left(\sigma_c - 1\right)\varepsilon_{L,t} L_t(h)^{\sigma_l} = -\lambda_t(h) \frac{W_t^m}{P_t}. \tag{3.6}$$

Writing $\mu_t(h)$ the Lagrange multiplier associated to the capital accumulation function one gets:

$$\begin{aligned} p_{I,t}\lambda_t(h) = \mu_t(h)\varepsilon_{I,t}&\left[1 - \mathcal{S}\left(\frac{I_t(h)}{I_{t-1}(h)}\right) - \frac{I_t(h)}{I_{t-1}(h)}\mathcal{S}'\left(\frac{I_t(h)}{I_{t-1}(h)}\right)\right] \\ +\beta \mathbb{E}_t &\left[\mu_{t+1}(h)\varepsilon_{I,t+1}\left(\frac{I_{t+1}(h)}{I_t(h)}\right)^2 \mathcal{S}'\left(\frac{I_{t+1}(h)}{I_t(h)}\right)\right] \end{aligned} \tag{3.7}$$

$$\mu_t(h)\delta'\left(z_t(h)\right) = \lambda_t(h) r_t^k \tag{3.8}$$

and at last:

$$\mu_t(h) = \beta \mathbb{E}_t \Big[\mu_{t+1}(h) \Big(1 - \delta(z_{t+1}(h)) \Big) + \lambda_{t+1}(h) r^k_{t+1} z_{t+1}(h) \Big] \tag{3.9}$$

Given the symmetrical nature of the solution for the household's problem, we get the following aggregated relationships:

$$(C_t - \eta C_{t-1})^{-\sigma_c} \exp \left\{ \varepsilon_{L,t} \frac{\sigma_c - 1}{1 + \sigma_l} L_t^{1+\sigma_l} \right\} = \lambda_t \tag{3.10}$$

$$\lambda_t = \beta \varepsilon_{B,t} R_t \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\pi_{t+1}} \right] \tag{3.11}$$

$$u_t \varepsilon_{L,t} \left(\sigma_c - 1 \right) L_t^{\sigma_l} = -\lambda_t \frac{W_t^m}{P_t} \tag{3.12}$$

$$\frac{p_{I,t}}{\varepsilon_{I,t}} = Q_t \left[1 - \mathcal{S} \left(\frac{I_t}{I_{t-1}} \right) - \frac{I_t}{I_{t-1}} \mathcal{S}' \left(\frac{I_t}{I_{t-1}} \right) \right] + \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} Q_{t+1} \frac{\varepsilon_{I,t+1}}{\varepsilon_{I,t}} \left(\frac{I_{t+1}}{I_t} \right)^2 \mathcal{S}' \left(\frac{I_{t+1}}{I_t} \right) \right] \tag{3.13}$$

$$Q_t \delta' \left(z_t \right) = r_t^k \tag{3.14}$$

$$Q_t = \beta \mathbb{E}_t \left[\frac{\lambda_{t+1}}{\lambda_t} \left(Q_{t+1} \Big(1 - \delta(z_{t+1}) \Big) + r^k_{t+1} z_{t+1} \right) \right] \tag{3.15}$$

Here, $Q_t \equiv {}^{\mu_t}/_{\lambda_t}$ is Tobin's Q.

3.2 Production

3.2.1 Final good producers

Producers of final good, Y_t, operate in a perfectly competitive environment, assembling a continuum of diversified intermediary goods written $Y_t(\iota)$ with $\iota \in [0, 1]$. They have access to a unique constant return aggregation technology as in **?**, implicitly defined by

$$\int_0^1 \mathscr{G}_f\left(\frac{Y_t(\iota)}{Y_t}\right)\mathrm{d}\iota = 1 \tag{3.16}$$

where $\mathscr{G}_f$ is a strictly increasing concave function such that $\mathscr{G}_f(1) = 1$. We follow **?** or **?** and adopt the following functional form for this aggregation function:

$$\mathscr{G}_f(x) = \frac{\theta_f(1+\psi_f)}{(1+\psi_f)(\theta_f(1+\psi_f)-1)}\Big[(1+\psi_f)x-\psi_f\Big]^{\frac{(1+\psi_f)\theta_f-1}{(1+\psi_f)\theta_f}} - \left[\frac{\theta_f(1+\psi_f)}{(1+\psi_f)(\theta_f(1+\psi_f)-1)}-1\right] \tag{3.17}$$

Parameter ψ_f characterize the curvature of the demand function.

The producer of final good chooses the quantity of intermediary goods ι so as to maximize her real profit:

$$Y_t - \int_0^1 \frac{P_t(\iota)}{P_t} Y_t(\iota)\mathrm{d}\iota$$

under the technological constraints (3.16) and (3.17). As the aggregation function is homogeneous of degree one, it is equivalent to minize cost per unit with respect to the relative demand of intermediary good ι under the technological constraint.

The first order condition of optimality determines the demand for intermediary good ι:

$$\frac{Y_t(\iota)}{Y_t} = \frac{1}{1+\psi_f}\left[\left(\frac{P_t(\iota)/P_t}{\Theta_t}\right)^{-(1+\psi_f)\theta_f} + \psi_f\right] \tag{3.18}$$

where Θ_t is the Lagrange multiplier associated with the technological constraints (3.16) and (3.17) for the representative firm. Substituting (3.18) in the technological constraints, one gets the following expression for the Lagrange multiplier:

$$\Theta_t = \left(\int_0^1 \left(\frac{P_t(\iota)}{P_t}\right)^{1-\theta_f(1+\psi_f)}\mathrm{d}\iota\right)^{\frac{1}{1-\theta_f(1+\psi_f)}} \tag{3.19}$$

The price elasticity of demand is given by:

$$\epsilon\left(\widetilde{Y}_t(\iota)\right) = -\frac{\mathscr{G}'\left(\widetilde{Y}_t(\iota)\right)}{\widetilde{Y}_t(\iota)\mathscr{G}''\left(\widetilde{Y}_t(\iota)\right)}$$

and, with particular aggregation function adpopted in this study,

$$\epsilon\left(\widetilde{Y}_t(\iota)\right) = \theta_f\left[1+\psi_f - \frac{\psi_f}{\widetilde{Y}_t(\iota)}\right]$$

When ψ_f is equal to zero, we get back to the more usual case of the CES aggregator of Dixit-Stiglitz **?** with a price elasticity of demand equal to θ_f. More generally, one remarks that demand is more sensitive to price when the level of demand is important if and only if parameter ψ_f is positive. We expect therefore to obtain a negative value for this parameter.

Finally, as the final good sector is perfectly competitive, profit for the representative firm must be zero and we derive the aggregate price index:

$$P_t = \frac{\psi_f}{1+\psi_f}\int_0^1 P_t(\iota)\mathrm{d}\iota + \frac{1}{1+\psi_f}\left(\int_0^1 P_t(\iota)^{1-(1+\psi_f)\theta_f}\mathrm{d}\iota\right)^{\frac{1}{1-(1+\psi_f)\theta_f}} \tag{3.20}$$

3.2.2 Intermediary goods producers

A continuum of firms $\iota \in [0,1]$ in monolistic competition produce intermediary goods for the producers of the final good. These firms have all access to the same Cobb–Douglas technology in to transform physical capital and labor in differentiated intermediary goods:

$$Y_t(\iota) = \left(K_t^d(\iota)\right)^{\alpha}\left(A_t L_t^d(\iota)\right)^{1-\alpha} \tag{3.21}$$

where $K_t^d(\iota)$ and $L_t^d(\iota)$ are demands of intermediary good firm ι for physical capital, and labor, respectively; A_t is technical progress, neutral in Harrod sense. The latter term is further decomposed in a trend component $\mathcal{A}_{T,t}$ and a cyclical on $\mathcal{A}_{C,t}$. We have then,

$$\Delta \log \mathcal{A}_{T,t} \sim \mathrm{AR}(1) \text{ stationary with mean } \log(1+g) \tag{3.22a}$$

$$\log \mathcal{A}_{C,t} \sim \mathrm{AR}(1) \text{ stationary with zero mean.} \tag{3.22b}$$

Each intermediary firm $\iota \in [0,1]$ buys freely its production factors on competitive markets taking their price as given. The firm $\iota \in [0,1]$ decides upon the mix of physical capital ($K_t^d(\iota)$) and labor ($L_t^d(\iota)$) so as to minimize its cost, $r_t^k K_t^d(\iota) + w_t L_t^d(\iota)$, under the technological constraint (3.21). The firm optimal behavior on the factor markets is summarized by the following factor prices frontier:

$$\frac{w_t L_t^d(\iota)}{r_t^k K_t^d(\iota)} = \frac{1-\alpha}{\alpha} \tag{3.23}$$

where $w_t \equiv W_t/P_t$ is the real wage. The ratio of capital to labor is invariant across firms. Using the factor prices frontier, we rewrite the total cost of firm ι as a function of the stock of capital:

$$CT_t(\iota) = \frac{r_t^k K_t^d(\iota)}{\alpha}$$

On the other hand, as the returns of scale is constant, we know that the total cost can also be written as

$$CT_t(\iota) = mc_t(\iota) Y_t(\iota)$$

where $mc_t(\iota)$ is the real marginal cost. We derive then the following expression for the marginal cost of firm ι:

$$mc_t(\iota) = A_t^{\alpha-1} \left(\frac{r_t^k}{\alpha}\right)^{\alpha} \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha} \equiv mc_t \tag{3.24}$$

Again, marginal cost doesn't depend on the size of the firm and is constant across firms.

The nominal profit of a firm that offers price $\mathcal{P}$ at date t id given by:

$$\Pi_t(\mathcal{P}) = \left(\varepsilon_{y,t}\frac{\mathcal{P}}{P_t} - mc_t\right)\left[\left(\frac{\mathcal{P}}{P_t}\right)^{-(1+\psi_f)\theta_f}\left(\int_0^1\left(\frac{P_t(\iota)}{P_t}\right)^{1-(1+\psi_f)\theta_f}\mathrm{d}f\right)^{\frac{(1+\psi_f)\theta_f}{1-(1+\psi_f)\theta_f}} + \psi_f\right]\frac{P_tY_t}{1+\psi_f}$$

where $\log\varepsilon_{y,t}$, an zero mean AR stationary process, is a shock to the sales of the firm.

Firm ι has market power but can't decide of the its optimal price in each period. Following a Calvo scheme, at each date, the firm receives a signal telling it whether it can revise its price $P_t(\iota)$ in an optimal manner or not. There is a probability ξ_p that the firm can't revise its price in a given period. In such a case, the firm follows the following rule:

$$P_t(\iota) = [\bar{\pi}_t]^{\gamma_p}\left[\frac{P_{t-1}}{P_{t-2}}\right]^{1-\gamma_p} P_{t-1}(\iota) = \Gamma_t P_{t-1}(\iota) \tag{3.25}$$

where $\bar{\pi}_t$ si the inflation target of the monetary authorities. More generally, we write

$$\Gamma_{t+j,t} = \left(\prod_{h=0}^{j-1}\bar{\pi}_{t+h}\right)^{\gamma_p}\left(\prod_{h=0}^{j-1}\pi_{t+h}\right)^{1-\gamma_p} = \Gamma_{t+1}\Gamma_{t+2}\dots\Gamma_{t+j}$$

the growth factor of the price of a firm that doesn't receive a favorable signal during j successive periods(for $j=0$ we have $\Gamma_{t,t}=1$; for $j=1$, we have $\Gamma_{t+1,t}=\Gamma_{t+1}$). When the firm ι receives a positive signal (with probability $1-\xi_p$), it chooses price $P_t(\iota)$ that maximizes its profit.

Let $\widetilde{\mathscr{V}}_t$ be the value of a firm that receives a positive signal in period t and $\mathscr{V}_t(P_{t-1}(\iota))$ the value of a firm that receives a negative signal. As a firm that receives a negative signal follows simply the *ad hoc* pricing rule (3.25), its value at time t depends only on $P_{t-1}(\iota)$. For a firm that receives a positive signal, its value at period t is

$$\widetilde{\mathcal{V}}_t = \max_{\mathbf{P}}\left\{\Pi_t\Big(\mathbf{P}\Big) + \beta\mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\Big((1-\xi_p)\widetilde{\mathcal{V}}_{t+1} + \xi_p\mathcal{V}_{t+1}(\mathbf{P})\Big)\right]\right\} \tag{3.26}$$

where Λ_t is the Lagrange multiplier of the budget constraint of the representative household and $P_t\Lambda_t = \lambda_t$. Let $P^\star$ be the optimal price choosen by the firm that can re–optimize.

The value of a firm that can't re–optimize is

$$\mathcal{V}_t\big(P_{t-1}(\iota)\big) = \Pi_t\Big(\Gamma_t P_{t-1}(\iota)\Big) + \beta\mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\Big((1-\xi_p)\widetilde{\mathcal{V}}_{t+1} + \xi_p\mathcal{V}_{t+1}\big(\Gamma_t P_{t-1}(\iota)\big)\Big)\right] \tag{3.27}$$

The first order condition and the envelope theorem give

$$\Pi'_t\big(P^\star\big) + \beta\xi_p\mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\mathcal{V}'_{t+1}\big(P^\star\big)\right] = 0 \tag{3.28a}$$

$$\frac{\mathcal{V}'_t\big(P_{t-1}(\iota)\big)}{\Gamma_t} = \Pi'_t\big(\Gamma_t P_{t-1}(\iota)\big) + \beta\xi_p\mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\mathcal{V}'_{t+1}\big(\Gamma_t P_{t-1}(\iota)\big)\right] \tag{3.28b}$$

with the derivative of profit at $\mathcal{P}$:

$$\begin{aligned}\Pi'_t\big(\mathcal{P}\big) \quad &= \quad \varepsilon_{y,t}\frac{1-\theta_f(1+\psi_f)}{1+\psi_f}\left(\frac{\mathcal{P}}{P_t}\right)^{-(1+\psi_f)\theta_f}\Theta_t^{(1+\psi_f)\theta_f}Y_t \\ &+\theta_f\left(\frac{\mathcal{P}}{P_t}\right)^{-(1+\psi_f)\theta_f-1}\Theta_t^{(1+\psi_f)\theta_f}mc_tY_t + \frac{\psi_f}{1+\psi_f}\varepsilon_{y,t}Y_t\end{aligned} \tag{3.29}$$

Let's write temporarily, in order to simplify notations, $\mathcal{P}$, the price inherited from the past. One can rewrite, one period ahead

$$\mathcal{V}'_{t+1}\big(\mathcal{P}\big) = \Gamma_{t+1,t}\Pi'_{t+1}\big(\Gamma_{t+1,t}\mathcal{P}\big) + \beta\xi_p\Gamma_{t+1,t}\mathbb{E}_{t+1}\left[\frac{\Lambda_{t+2}}{\Lambda_{t+1}}\mathcal{V}'_{t+2}\big(\Gamma_{t+1,t}\mathcal{P}\big)\right]$$

Iterating toward the future and applying conditional expectation at time t, one gets

$$\mathbb{E}_t\left[\mathcal{V}'_{t+1}\big(\mathcal{P}\big)\right] = \mathbb{E}_t\left[\sum_{j=0}^{\infty}(\beta\xi_p)^j\Gamma_{t+1+j,t}\frac{\Lambda_{t+1+j}}{\Lambda_{t+1}}\Pi'_{t+1+j}\big(\Gamma_{t+1+j,t}\mathcal{P}\big)\right]$$

By substitution ($\mathcal{P} = P^\star$) inf the first order condition, one gets the following condition for the price choosen by the firm that gets a positive signal:

$$\mathbb{E}_t\left[\sum_{j=0}^{\infty}(\beta\xi_p)^j\Gamma_{t+j,t}\frac{\Lambda_{t+j}}{\Lambda_t}\Pi'_{t+j}\big(\Gamma_{t+j,t}P_t^\star\big)\right] = 0 \tag{3.30}$$

One can get a more explicit expression for the price that satisfies equation (3.30). Substituting in this equation the expression of marginal profit (3.29) and dividing by $P_t^{\star\,-(1+\psi_f)\theta_f}$ one gets:

$$\frac{P_t^\star}{P_t} = \frac{\theta_f(1+\psi_f)}{\theta_f(1+\psi_f)-1}\frac{\mathscr{Z}_{1,t}}{\mathscr{Z}_{2,t}} + \frac{\psi_f}{\theta_f(1+\psi_f)-1}\left(\frac{P_t^\star}{P_t}\right)^{1+(1+\psi_f)\theta_f}\frac{\mathscr{Z}_{3,t}}{\mathscr{Z}_{2,t}} \tag{3.31}$$

with

$$\mathscr{Z}_{1,t} = \mathbb{E}_t\sum_{i=0}^{\infty}(\beta\xi_p)^j\lambda_{t+j}\left(\frac{\Gamma_{t+j}}{P_{t+j}/P_t}\right)^{-(1+\psi_f)\theta_f}\Theta_{t+j}^{(1+\psi_f)\theta_f}mc_{t+j}Y_{t+j} \tag{3.32a}$$

$$\mathscr{Z}_{2,t} = \mathbb{E}_t\sum_{i=0}^{\infty}(\beta\xi_p)^j\lambda_{t+j}\varepsilon_{y,t+j}\left(\frac{\Gamma_{t+j}}{P_{t+j}/P_t}\right)^{1-(1+\psi_f)\theta_f}\Theta_{t+j}^{(1+\psi_f)\theta_f}Y_{t+j} \tag{3.32b}$$

$$\mathscr{Z}_{3,t} = \mathbb{E}_t\sum_{i=0}^{\infty}(\beta\xi_p)^j\lambda_{t+j}\varepsilon_{y,t+j}\frac{\Gamma_{t+j}}{P_{t+j}/P_t}Y_{t+j} \tag{3.32c}$$

writing P_{t+j}/P_t, the inflation factor between t and $t+j$, can be written equivalently $\Pi_{i=1}^{j}\pi_{t+i}$, and we can represent variables $\mathscr{Z}_{1,t}$, $\mathscr{Z}_{2,t}$ and $\mathscr{Z}_{3,t}$ in recursive form:

$$\mathscr{Z}_{1,t} = \widehat{\lambda}_t mc_t\Theta_t^{(1+\psi_f)\theta_f}\widehat{Y}_t + \beta\xi_p\mathbb{E}_t\left[\left(\frac{\pi_{t+1}}{\pi_{t-1}^{\gamma_p}\bar{\pi}_t^{1-\gamma_p}}\right)^{(1+\psi_f)\theta_f}\mathscr{Z}_{1,t+1}\right] \tag{3.33a}$$

$$\mathscr{Z}_{2,t} = \widehat{\lambda}_t\varepsilon_{y,t}\Theta_t^{(1+\psi_f)\theta_f}\widehat{Y}_t + \beta\xi_p\mathbb{E}_t\left[\left(\frac{\pi_{t+1}}{\pi_{t-1}^{\gamma_p}\bar{\pi}_t^{1-\gamma_p}}\right)^{(1+\psi_f)\theta_f-1}\mathscr{Z}_{2,t+1}\right] \tag{3.33b}$$

$$\mathscr{Z}_{3,t} = \widehat{\lambda}_t\varepsilon_{y,t}\widehat{Y}_t + \beta\xi_p\mathbb{E}_t\left[\left(\frac{\pi_{t-1}^{\gamma_p}\bar{\pi}_t^{1-\gamma_p}}{\pi_{t+1}}\right)\mathscr{Z}_{3,t+1}\right] \tag{3.33c}$$

Writing $\vartheta_{f,t} \equiv \int_0^1 \frac{P_t(\iota)}{P_t}\mathrm{d}\iota$ can be written in recursive form:

$$\vartheta_{f,t} = (1-\xi_p)\frac{P_t^\star}{P_t} + \xi_p\frac{\bar{\pi}_t^{1-\gamma_p}\pi_{t-1}^{\gamma_p}}{\pi_t}\vartheta_{f,t-1} \tag{3.34}$$

we can finally write the equation (3.20) equaivalently as:

$$\frac{\psi_f\vartheta_{f,t}}{1+\psi_f} + \frac{\Theta_t}{1+\psi_f} = 1 \tag{3.35}$$

In the end, inflation dynamics are characterized by equations (3.35), (3.34), (3.31), (3.33a), (3.33b), (3.33c).

3.3 Labor

Homogeneous labor $L_t = \int_0^1 L_t(h)\mathrm{d}h$ provided by the households is differentiated by a continuum of unions, $\varsigma \in [0,1]$. We have then $L_t = \int_0^1 l_t(\varsigma)\mathrm{d}\varsigma$. Unions sell differentiated labor, $l_t(\varsigma)$, to an employment agency that aggregate different types of labor to offer it as input to the firms of the intermediary good sector. Unions have monopolistic power and the employment agency operates in a perfectly competitive manner.

3.3.1 Employment agency

It aggregates lagor $l_t(\varsigma)$ provided by unions with an aggregation function as in **?**, defined implicitly by

$$\int_0^1 \mathscr{G}_s\left(\frac{l_t(\varsigma)}{\mathcal{L}_t}\right)\mathrm{d}\varsigma = 1 \tag{3.36}$$

where $\mathscr{G}_s$ is a strictly increasing concave function such that $\mathscr{G}_s(1) = 1$. We use the following specification:

$$\mathscr{G}_s(x) = \frac{\theta_s(1+\psi_s)}{(1+\psi_s)(\theta_s(1+\psi_s)-1)}\Big[(1+\psi_s)x-\psi_s\Big]^{\frac{(1+\psi_s)\theta_s-1}{(1+\psi_s)\theta_s}} - \left[\frac{\theta_s(1+\psi_s)}{(1+\psi_s)(\theta_s(1+\psi_s)-1)}-1\right] \tag{3.37}$$

This function is a generalization of the Dixit–Stiglitz aggregator **?** that is a particular case when $\psi_s = 0$. The employment agency chooses the relative quantity of labor of type ς such as minimizing the cost of production by unit of homogenous labor, $\frac{W_t(\varsigma)}{W_t}\frac{l_t(\varsigma)}{\mathcal{L}_t}$, under the technological constraints (3.36) and (3.37). The first order condition associated to the optimization program of the employment agency determines its demand of differentiated labor[3] ς:

$$\frac{l_t(\varsigma)}{\mathcal{L}_t} = \frac{1}{1+\psi_s}\left[\left(\frac{W_t(\varsigma)/W_t}{\Upsilon_t}\right)^{-(1+\psi_s)\theta_s} + \psi_s\right] \tag{3.38}$$

[3] In order to save in notations, we don't make a difference between the demand of the employment agency and the supply by the unions.

where Υ_t is the Lagrance multiplier associated with the technological constraints (3.36) and (3.37. Substituting (3.38) in the technological constraint, one gets the following expression for the Lagrange multiplier

$$\Upsilon_t = \left(\int_0^1 \left(\frac{W_t(\varsigma)}{W_t} \right)^{1-\theta_s(1+\psi_s)} \mathrm{d}\varsigma \right)^{\frac{1}{1-\theta_s(1+\psi_s)}} \tag{3.39}$$

As the employment agency behaves in a competitive manner, its profit is zero and we get the aggregate wage as

$$W_t = \frac{\psi_s}{1+\psi_s} \int_0^1 W_t(\varsigma)\mathrm{d}\varsigma + \frac{1}{1+\psi_s} \left(\int_0^1 W_t(\varsigma)^{1-(1+\psi_s)\theta_s} \mathrm{d}\varsigma \right)^{\frac{1}{1-(1+\psi_s)\theta_s}} \tag{3.40}$$

3.3.2 Unions

Unions supply differentiated labor services from the homogeneous labor supply from the households. Unions have market power because of this differentiation of labor services. We write the profit of a union offering wage $W_t(\varsigma)$ and $l_t(\varsigma)$ units of labor:

$$\mathscr{S}_t\left(W_t(\varsigma)\right)) = \left(\varepsilon_{l,t} W_t(\varsigma) - W_t^m\right) l_t(\varsigma)$$

where $\log \varepsilon_{l,t}$, is an exogenous shock on union's gains. It is an AR(1) process with zero mean. For a given demand, when $\varepsilon_{l,t} = 1$, the profit of the union is given by the difference between the wage asked to the employment agency and the wage paid to the household. By substitution in the demand function of the employment agency (3.38):

$$\mathscr{S}_t\left(W_t(\varsigma)\right)) = \left(\varepsilon_{l,t} W_t(\varsigma) - W_t^m\right) \frac{1}{1+\psi_s} \left[\left(\frac{W_t(\varsigma)/W_t}{\Upsilon_t} \right)^{-(1+\psi_s)\theta_s} + \psi_s \right] \mathcal{L}_t \tag{3.41}$$

Each union is subject to a Calvo lottery. In each period, a union can adjust the wage $W_t(\varsigma)$ in an optimal manner with probability ξ_w. In this case, the union chooses wage $W_t^\star$ that maximizes its profit knowing that in the future it may not have the opportunity to readjust the wage for some periods. When the lottery draw is negative for the union (with probability $1-\xi_w$), it adjusts wages according to the following *ad hoc* rule:

$$W_t(\varsigma) = \frac{\mathcal{A}_{T,t}}{\mathcal{A}_{T,t-1}} \bar{\pi}_{t-1}^{\gamma_w} \pi_{t-1}^{1-\gamma_w} W_{t-1}(\varsigma) \tag{3.42}$$

We write $\Omega_t = \left(\mathcal{A}_{T,t}/\mathcal{A}_{T,t-1}\right) \bar{\pi}_{t-1}^{\gamma_w} \pi_{t-1}^{1-\gamma_w} \equiv \Omega_{t,t-1}$ the growth factor of nominal wage asked by the union ς at date t when this one doesn't have the opportunity of revising it in an optimal manner. In this case, the union changes the wage by indexing it on (*i*) a convex mix of the inflation target of the monetary authority and of past inflation and (*ii*) the efficiency growth in the intermediary goods sector. We write

$$\Omega_{t+j,t} = \frac{\mathcal{A}_{T,t+j}}{\mathcal{A}_{T,t}} \left(\prod_{h=0}^{j-1} \bar{\pi}_{t+h} \right)^{\gamma_w} \left(\prod_{h=0}^{j-1} \pi_{t+h} \right)^{1-\gamma_w} = \Omega_{t+1}\Omega_{t+2} \dots \Omega_{t+j}$$

the growth factor of the wage of a a union that gets negative signals during the the next j periods (for $j = 0$, we have $\Omega_{t,t} = 1$, for $j = 1$, we have $\Omega_{t+1,t} = \Omega_{t+1}$).

Let $\widetilde{\mathscr{U}_t}$ be the value of a union that receives a positive signal at date t and $\mathscr{U}_t(W_{t-1}(\varsigma))$ the value of a union that receives the negative signal. In the latter case, the union follows simply the *ad hoc* rule (3.42), this explains why its value at date t depends upon $W_{t-1}(\varsigma)$. On the opposite, the optimization program of a union that receives a positive signal is purely turn towards the future. As unions have the same expectations about the future, they all choose the same optimal wage ($W_t^\star$). More formally, the value at date t of a union that receives a positive signal is

$$\widetilde{\mathcal{U}}_t = \max_{\mathbf{W}} \left\{ \mathscr{S}_t\big(\mathbf{W}\big) + \beta \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \left((1-\xi_w)\widetilde{\mathcal{U}}_{t+1} + \xi_w \mathcal{U}_{t+1}\big(\mathbf{W}\big) \right) \right] \right\} \tag{3.43}$$

where Λ_t is the Lagrange multiplier associated to the nominal budget constraint of the representative household.

The value of a union that receives a negative signal is

$$\mathcal{U}_t\big(W_{t-1}(\varsigma)\big) = \mathscr{S}_t\Big(\Gamma_t W_{t-1}(\varsigma)\Big) + \beta \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \left((1-\xi_w)\widetilde{\mathcal{U}}_{t+1} + \xi_w \mathcal{U}_{t+1}\big(\Gamma_t W_{t-1}(\varsigma)\big) \right) \right] \tag{3.44}$$

The first order condition and the application of the envelop theorem give

$$\mathscr{S}_t'\big(W_t^\star\big) + \beta\xi_w \mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\mathcal{U}_{t+1}'\big(W_t^\star\big)\right] = 0 \tag{3.45a}$$

$$\frac{\mathcal{U}_t'\big(W_{t-1}(\varsigma)\big)}{\Gamma_t} = \mathscr{S}_t'\big(\Gamma_t W_{t-1}(\varsigma)\big) + \beta\xi_p \mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\mathcal{U}_{t+1}'\big(\Gamma_t W_{t-1}(\varsigma)\big)\right] \tag{3.45b}$$

with the derivative of the union profit at $\mathcal{W}$:

$$\begin{aligned}\mathscr{S}_t'(\mathcal{W}) \quad &= \quad \varepsilon_{l,t}\frac{1-\theta_s(1+\psi_s)}{1+\psi_s}\left(\frac{\mathcal{W}}{W_t}\right)^{-(1+\psi_s)\theta_s}\Upsilon_t^{(1+\psi_s)\theta_s}\mathcal{L}_t \\ &+\theta_s\left(\frac{\mathcal{W}}{W_t}\right)^{-(1+\psi_s)\theta_s-1}\Upsilon_t^{(1+\psi_s)\theta_s}\frac{W_t^m}{W_t}\mathcal{L}_t + \frac{\psi_s}{1+\psi_s}\varepsilon_{l,t}\mathcal{L}_t\end{aligned} \tag{3.46}$$

Let's write temporarily, in order to simplify notations, $\mathcal{W}$, the price inherited from the past. One can rewrite, one period ahead

$$\mathcal{U}_{t+1}'(\mathcal{W}) = \Omega_{t+1,t}\mathscr{S}_{t+1}'\big(\Omega_{t+1,t}\mathcal{W}\big) + \beta\xi_p\Omega_{t+1,t}\mathbb{E}_{t+1}\left[\frac{\Lambda_{t+2}}{\Lambda_{t+1}}\mathcal{U}_{t+2}'\big(\Omega_{t+1,t}\mathcal{W}\big)\right]$$

iterating toward the future and applying conditional expectation gives

$$\mathbb{E}_t\left[\mathcal{U}_{t+1}'(\mathcal{W})\right] = \mathbb{E}_t\left[\sum_{j=0}^{\infty}(\beta\xi_p)^j\Omega_{t+1+j,t}\frac{\Lambda_{t+1+j}}{\Lambda_{t+1}}\mathscr{S}_{t+1+j}'\big(\Omega_{t+1+j,t}\mathcal{W}\big)\right]$$

Substituting in the first order condition (pour $\mathcal{W} = W_t^\star$), one gets the following condition for an optimal wage choice by a union that receives a positive signal:

$$\mathbb{E}_t\left[\sum_{j=0}^{\infty}(\beta\xi_p)^j\Omega_{t+j,t}\frac{\Lambda_{t+j}}{\Lambda_t}\mathscr{S}_{t+j}'\big(\Omega_{t+j,t}W_t^\star\big)\right] = 0 \tag{3.47}$$

The optimal wage $W_t^\star$ is the nominal wage that insures that the sum of current and expected discounted marginal profits are zero when the union can only revise nominal wages by using the *ad hoc* rule (3.42).

It is possible to obtain a recursive expression for multiplier Υ_t that appears in the expression for a union profit. Equation (3.39) can be written equivalently in the form

$$\Upsilon_t^{1-\theta_s(1+\psi_s)} = \int_0^1 \left(\frac{W_t(\varsigma)}{W_t}\right)^{1-\theta_s(1+\psi_s)} \mathrm{d}\varsigma$$

The wage offered by the union at date t appears under the integral sign. This price has been determined optimaly j periods before with probability $(1-\xi_w)\xi_w^j$. We can then rewrite the integral as

$$\Upsilon_t^{1-\theta_s(1+\psi_s)} = (1-\xi_w)\sum_{j=0}^{\infty} \xi_w^j \left(\frac{\Omega_{t,t-j} W_{t-j}^\star}{W_t}\right)^{1-\theta_s(1+\psi_s)}$$

where $W_{t-j}^\star$ is the optimal wage at date $t-j$. Finally, one can interpret the infinite sum as the solution of the following recursive equation:

$$\Upsilon_t^{1-\theta_s(1+\psi_s)} = (1-\xi_w)\left(\frac{W_t^\star}{W_t}\right)^{1-\theta_s(1+\psi_s)} + \xi_w \left(\frac{\Omega_{t,t-1}}{W_t/W_{t-1}}\right)^{1-\theta_s(1+\psi_s)} \Upsilon_{t-1}^{1-\theta_s(1+\psi_s)} \tag{3.48}$$

One can get a more explicit expression for the wage that satisfies equation (3.47). Substituting in this equation the expression for marginal profit (3.46) and dividing by $W_t^{\star\,-(1+\psi_s)\theta_s}$, one gets

$$\frac{w_t^\star}{w_t} = \frac{\theta_s(1+\psi_s)}{\theta_s(1+\psi_s)-1}\frac{\mathscr{H}_{1,t}}{\mathscr{H}_{2,t}} + \frac{\psi_s}{\theta_s(1+\psi_s)-1}\left(\frac{w_t^\star}{w_t}\right)^{1+(1+\psi_s)\theta_s}\frac{\mathscr{H}_{3,t}}{\mathscr{H}_{2,t}} \tag{3.49}$$

where $w_t^\star$ is the real wage obtained by the union at date t when it can adjust the nominal wage in an optimal manner and w_t the real nominal wage in the economy, with

$$\mathscr{H}_{1,t} = \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\xi_w)^j \lambda_{t+j} w_{t+j}^m \left(\frac{\Omega_{t+j}}{\frac{w_{t+j}}{w_t}\frac{P_{t+j}}{P_t}}\right)^{-(1+\psi_s)\theta_s} \Upsilon_{t+j}^{(1+\psi_s)\theta_s} \mathcal{L}_{t+j} \tag{3.50a}$$

$$\mathscr{H}_{2,t} = \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\xi_w)^j \lambda_{t+j} \varepsilon_{l,t+j} w_{t+j} \left(\frac{\Omega_{t+j}}{\frac{w_{t+j}}{w_t}\frac{P_{t+j}}{P_t}}\right)^{1-(1+\psi_s)\theta_s} \Upsilon_{t+j}^{(1+\psi_s)\theta_s} \mathcal{L}_{t+j} \tag{3.50b}$$

$$\mathscr{H}_{3,t} = \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\xi_w)^j \lambda_{t+j} \varepsilon_{l,t+j} w_{t+j} \frac{\Omega_{t+j}}{\frac{w_{t+j}}{w_t}\frac{P_{t+j}}{P_t}} \mathcal{L}_{t+j} \tag{3.50c}$$

Noticing that w_{t+j}/w_t, the growth factor of the real wage between t and $t+j$, can be equivalently written as $\Pi_{i=1}^{j} \varpi_{t+i}$ (ϖ_t is the growth factor of the real wage between t and $t-1$) and that we have

$$\Omega_{t+j,t} = (1+g)^j \left(\prod_{h=1}^{j} \mathscr{E}_{t+h} \right)^{\frac{1}{1-\rho_x}} \left(\prod_{h=0}^{j-1} \bar{\pi}_{t+h} \right)^{\gamma_w} \left(\prod_{h=0}^{j-1} \pi_{t+h} \right)^{1-\gamma_w},$$

we can finally represent variables $\mathscr{H}_{1,t}$, $\mathscr{H}_{2,t}$ and $\mathscr{H}_{3,t}$ in the recursive form

$$\mathscr{H}_{1,t} = \lambda_t w_t^m \mathcal{L}_t \Upsilon_t^{(1+\psi_s)\theta_s} + \beta \xi_w \mathbb{E}_t \left[\left(\frac{\varpi_{t+1} \pi_{t+1}}{(1+g) \mathscr{E}_{t+1}^{\frac{1}{1-\rho_x}} \pi_{t-1}^{\gamma_w} \bar{\pi}_t^{1-\gamma_w}} \right)^{(1+\psi_s)\theta_s} \mathscr{H}_{1,t+1} \right] \tag{3.51a}$$

$$\mathscr{H}_{2,t} = \lambda_t \varepsilon_{l,t} w_t \mathcal{L}_t \Upsilon_t^{(1+\psi_s)\theta_s} + \beta \xi_w \mathbb{E}_t \left[\left(\frac{\varpi_{t+1} \pi_{t+1}}{(1+g) \mathscr{E}_{t+1}^{\frac{1}{1-\rho_x}} \pi_{t-1}^{\gamma_w} \bar{\pi}_t^{1-\gamma_w}} \right)^{(1+\psi_s)\theta_s - 1} \mathscr{H}_{2,t+1} \right] \tag{3.51b}$$

$$\mathscr{H}_{3,t} = \lambda_t \varepsilon_{l,t} w_t \mathcal{L}_t + \beta \xi_w \mathbb{E}_t \left[\left(\frac{(1+g) \mathscr{E}_{t+1}^{\frac{1}{1-\rho_x}} \pi_{t-1}^{\gamma_w} \bar{\pi}_t^{1-\gamma_w}}{\varpi_{t+1} \pi_{t+1}} \right) \mathscr{H}_{3,t+1} \right] \tag{3.51c}$$

Noticing that $\vartheta_{s,t} \equiv \int_0^1 \frac{W_t(\varsigma)}{W_t} \mathrm{d}\varsigma$ can be written in the recursive form

$$\vartheta_{s,t} = (1-\xi_w) \frac{w_t^\star}{w_t} + \xi_w \frac{(1+g) \mathscr{E}_t^{\frac{1}{1-\rho_x}} \pi_{t-1}^{\gamma_w} \bar{\pi}_t^{1-\gamma_w}}{\varpi_t \pi_t} \vartheta_{s,t-1} \tag{3.52}$$

we can rewrite equation (3.40) as

$$\frac{\psi_s \vartheta_{s,t}}{1+\psi_s} + \frac{\Upsilon_t}{1+\psi_s} = 1 \tag{3.53}$$

In the end, wage dynamics are characterized by equations (3.45), (3.52), (3.48), (3.49), (3.51a), (3.51b), (3.51c).

3.4 Government and monetary authority

3.4.1 Fiscal policy

We assume that exogenous government expenditures $G_t = g_t Y_t$ are exactly financed by lump sum taxes:

$$T_t = P_t G_t.$$

3.4.2 Central Bank

We assume that the behavior of the central bank is adequately described by a simple Taylor rule:

$$R_t = R_{t-1}^{\rho_R}\left[R^\star\left(\frac{\pi_{t-1}}{\bar{\pi}_t}\right)^{r_\pi}\left(\frac{Y_t}{\mathscr{Y}_t}\right)^{r_Y}\right]^{1-\rho_R}\varepsilon_{R,t} \tag{3.54}$$

where $\bar{\pi}_t$ is the inflation target of the central bank, $\mathscr{Y}_t$ is a reference output level to be defined later, $\log \varepsilon_{R,t}$ is an AR(1) stationary process with zero mean.

3.5 General equilibrium

3.5.1 Price distortion

Prices in the intermediary good sector are heterogeneous. However, it is possible to show that this heterogeneity doesn't hinder aggregation. We know that the firms of this sector choose all the same mix of factor of production in the sense that the ratio of capital demand to labor demand is constant across firms (see equation 3.23). Expressing labor demand of firm ι as a function of its demand of physical capital, we can write the production of this firm

$$y_t(\iota) = \left(\frac{A_t L_t^d}{K_t^d}\right)^{1-\alpha} K_t^d(\iota)$$

When $K_t^d \equiv \int_0^1 K_t^d(\iota)\mathrm{d}\iota$ is aggregate demand for physical capital and $y_t \equiv \int_0^1 Y_t(\iota)\mathrm{d}\iota$ represents the sum of intermediary productions, we can write directly

$$y_t = \left(K_t^d\right)^\alpha \left(A_t L_t^d\right)^{1-\alpha}$$

The sum of intermediary productions is different from Y_t, because aggregation technology isn't linear. Integrating the demand function for good ι from the final good producers (3.18) over ι, we get

$$y_t = \Delta_{p,t} Y_t \tag{3.55}$$

with

$$\Delta_{p,t} \equiv \frac{1}{1+\psi_f} \int_0^1 \left(\left(\frac{P_t(\iota)/P_t}{\Theta_t} \right)^{-(1+\psi_f)\theta_f} + \psi_f \right) \mathrm{d}\iota \tag{3.56}$$

Price distortion can be written recursively in the following manner:

$$\Delta_{p,t} = \frac{1}{1+\psi_f} \Theta_t^{(1+\psi_f)\theta_f} \nabla_{p,t} + \frac{\psi_f}{1+\psi_f} \tag{3.57a}$$

$$\nabla_{p,t} = (1-\xi_p) \left(\frac{P_t^\star}{P_t} \right)^{-(1+\psi_f)\theta_f} + \xi_p \left(\frac{\bar{\pi}_t^{\gamma_p} \pi_{t-1}^{1-\gamma_p}}{\pi_t} \right)^{-(1+\psi_f)\theta_f} \nabla_{p,t-1} \tag{3.57b}$$

where the Lagrange multiplier Θ_t is defined recursively as well. Then, we have

$$\Delta_{p,t} Y_t = \left(K_t^d \right)^\alpha \left(A_t L_t^d \right)^{1-\alpha} \tag{3.58}$$

3.5.2 Wage distortion

Here, we show how to link aggregate labor supply by the households with aggregated labor supply by the employment agency to the firms of the intermediary good sector This link is affected by the heterogeneity of wages induced by their nominal rigidity. Integrating labor demand for type ς by the employment agency (3.38) over ς, we find directly

$$L_t = \Delta_{w,t} \mathcal{L}_t \tag{3.59}$$

with

$$\Delta_{w,t} \equiv \frac{1}{1+\psi_s} \int_0^1 \left(\left(\frac{W_t(\varsigma)/W_t}{\Upsilon_t} \right)^{-(1+\psi_s)\theta_s} + \psi_s \right) \mathrm{d}\varsigma \tag{3.60}$$

Wage distortion can be written in recursive form:

$$\Delta_{w,t} = \frac{1}{1+\psi_s}\Upsilon_t^{(1+\psi_s)\theta_s}\nabla_{w,t} + \frac{\psi_s}{1+\psi_s} \tag{3.61a}$$

$$\nabla_{w,t} = (1-\xi_w)\left(\frac{w_t^\star}{w_t}\right)^{-(1+\psi_s)\theta_s} + \xi_s\left(\frac{(1+g)\mathscr{E}_t^{\frac{1}{1-\rho_x}}\bar{\pi}_t^{\gamma_w}\pi_{t-1}^{1-\gamma_w}}{\varpi_t\pi_t}\right)^{-(1+\psi_s)\theta_s}\nabla_{w,t-1} \tag{3.61b}$$

where the Lagrange multiplier Υ_t is defined recursively in equation (3.48).

3.5.3 Dividends paid by intermediary good firms

Firms in the intermediary good sector interact in monopolistic competition and make profits that are paid to households in the form of dividends. The sum of nominal profits at date t is

$$\begin{aligned}
\Pi_t &= \int_0^1 \Pi_t(\iota)\mathrm{d}\iota \\
&= \int_0^1 \left(\varepsilon_{y,t}\frac{P_t(\iota)}{P_t} - mc_t\right)P_tY_t(\iota)\mathrm{d}\iota \\
&= \varepsilon_{y,t}\int_0^1 P_t(\iota)Y_t(\iota)\mathrm{d}\iota - P_tmc_ty_t \\
&= \varepsilon_{y,t}P_tY_t - \left(P_tr_t^k\int_0^1 K_t^d(\iota)\mathrm{d}\iota + W_t\int_0^1 L_t^d(\iota)\mathrm{d}\iota\right) \\
&= P_t\left(\varepsilon_{y,t}Y_t - r_t^kK_t^d - w_tL_t^d\right)
\end{aligned}$$

As households own the firms, the profits are paid to them. The repartition of these profits between the households in undetermined in general equilibrium, but we know that

$$\int_0^1 \mathscr{D}_{1,t}(h)\mathrm{d}h = P_t\left(\varepsilon_{y,t}Y_t - r_t^kK_t^d - w_tL_t^d\right) \tag{3.62}$$

3.5.4 Dividends paid by the unions

In the same way, we can compute aggregate nominal profit of the unions at date t. These profits as well are paid to the households. We have

$$\begin{aligned}
\mathscr{S}_t &= \int_0^1 \mathscr{S}_t(\varsigma)\mathrm{d}\varsigma \\
&= \int_0^1 \left(\varepsilon_{l,t}W_t(\iota) - W_t^m\right) l_t(\varsigma)\mathrm{d}\varsigma \\
&= \varepsilon_{l,t}\int_0^1 W_t(\varsigma)l_t(\varsigma)\mathrm{d}\varsigma - W_t^m L_t \\
&= \varepsilon_{l,t}W_t\mathcal{L}_t - W_t^m L_t
\end{aligned}$$

and, then,

$$\int_0^1 \mathscr{D}_{2,t}(h)\mathrm{d}h = \varepsilon_{l,t}W_t\mathcal{L}_t - W_t^m L_t \tag{3.63}$$

3.5.5 Equilibrium in factor markets and in bond markets

In general equilibrium, labor supply form the employment agency equals aggregate labor demand by firms of the intermediary good market. In the same way, aggregate supply of physical capital by the households equals aggregate demand from these firms. In formal terms,

$$\mathcal{L}_t \equiv \Delta_{w,t}^{-1}\int_0^1 L_t(h)\mathrm{d}h = \int_0^1 L_t^d(\iota)\mathrm{d}\iota \equiv L_t^d \tag{3.64}$$

$$\widetilde{K}_t \equiv \int_0^1 z_t(h)K_{t-1}(h)\mathrm{d}h = \int_0^1 K_t^d(\iota)\mathrm{d}\iota \equiv K_t^d \tag{3.65}$$

Finally, aggregate demand for bonds must be zero, as we assume a close economy and no government debt.

$$\int_0^1 B_t(h)\mathrm{d}h = 0 \tag{3.66}$$

3.5.6 Equilibrium on the good market

By summing the budget constraints of the households (3.2) over $h \in [0,1]$ and by substituting the equilibrium conditions on the bond market, the definition of aggregate dividends and the budget constraint of the government, we get

$$P_tG_t + P_tC_t + p_{I,t}P_tI_t = W_t^m L_t + P_t r_t^K z_t K_{t-1} + P_t\left(\varepsilon_{y,t}Y_t - r_t^k K_t^d - w_t L_t^d\right)$$
$$+ \varepsilon_{lt}W_t\mathcal{L}_t - W_t^m L_t$$

After simplification and knowing that the factor markets are in equilibrium, we obtain

$$G_t + C_t + p_{I,t}I_t = \varepsilon_{y,t}Y_t + (\varepsilon_{l,t} - 1)\, w_t\mathcal{L}_t \tag{3.67a}$$

or equivalently

$$G_t + C_t + p_{I,t}I_t = \varepsilon_{y,t}\Delta_{p,t}^{-1}y_t + (\varepsilon_{l,t} - 1)\,\Delta_{w,t}^{-1}w_tL_t \tag{3.67b}$$

3.6 Long run

The economy described in the model is growing at a constant rate in the long run, in absence of shocks. In order to compute a local approximation in order to solve it and later to estimate it, we need first to make it stationary.

In this model, there is a single source of growth in the long run: the Harrodian technical change that is characterized by a stochastic trend (with a deterministic component) and affects most of the real variables. In order to stationarize the model, we divide all non–stationary variables by $\mathcal{A}_{T,t}$ or a power of this variable. In this section we present the stationarized model and then its stationary state.

3.6.1 Stationary version of the model

On the household side, we write $C_t \equiv \widehat{C}_t\mathcal{A}_{T,t}$, $\lambda_t = \widehat{\lambda}_t\mathcal{A}_{T,t}^{-\sigma_c}$, $w_t^m = \widehat{w}_t^m\mathcal{A}_{T,t}$, $I_t \equiv \widehat{I}_t\mathcal{A}_{T,t}$. In order to make estimation (or calibration) of the long run level of $\varepsilon_{L,t}$ (*ie* the scale factor that lets obtain the desired long run level for labor hours), we redefine the utility function, replacing L_t by ${}^{L_t}/_{L}$

where L is the long run level of labor hours.

$$\left(\widehat{C}_t - \eta \frac{\mathcal{A}_{T,t-1}}{\mathcal{A}_{T,t}} \widehat{C}_{t-1}\right)^{-\sigma_c} \exp\left\{\varepsilon_{L,t} \frac{\sigma_c - 1}{1+\sigma_l} \left(\frac{L_t}{L}\right)^{1+\sigma_l}\right\} = \widehat{\lambda}_t \tag{3.68}$$

$$\widehat{\lambda}_t = \beta \varepsilon_{B,t} R_t \mathbb{E}_t \left[\left(\frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}}\right)^{-\sigma_c} \frac{\widehat{\lambda}_{t+1}}{\pi_{t+1}}\right] \tag{3.69}$$

$$\left[\widehat{C}_t - \eta \frac{\mathcal{A}_{T,t-1}}{\mathcal{A}_{T,t}} \widehat{C}_{t-1}\right]^{1-\sigma_c} \frac{\varepsilon_{L,t}}{L} \exp\left\{\varepsilon_{L,t} \frac{\sigma_c - 1}{1+\sigma_l} \left(\frac{L_t}{L}\right)^{1+\sigma_l}\right\} \left(\frac{L_t}{L}\right)^{\sigma_l} = \widehat{\lambda}_t \widehat{w}_t^m \tag{3.70}$$

$$\begin{aligned} \frac{p_{I,t}}{\varepsilon_{I,t}} = Q_t & \left[1 - \mathcal{S}\left(\frac{\widehat{I}_t}{\widehat{I}_{t-1}} \frac{\mathcal{A}_{T,t}}{\mathcal{A}_{T,t-1}}\right) - \frac{\widehat{I}_t}{\widehat{I}_{t-1}} \frac{\mathcal{A}_{T,t}}{\mathcal{A}_{T,t-1}} \mathcal{S}'\left(\frac{\widehat{I}_t}{\widehat{I}_{t-1}} \frac{\mathcal{A}_{T,t}}{\mathcal{A}_{T,t-1}}\right)\right] \\ & + \beta \mathbb{E}_t \left[\frac{\widehat{\lambda}_{t+1}}{\widehat{\lambda}_t} \left(\frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}}\right)^{-\sigma_c} Q_{t+1} \frac{\varepsilon_{I,t+1}}{\varepsilon_{I,t}} \left(\frac{\widehat{I}_{t+1}}{\widehat{I}_t} \frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}}\right)^2 \mathcal{S}'\left(\frac{\widehat{I}_{t+1}}{\widehat{I}_t} \frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}}\right)\right] \end{aligned} \tag{3.71}$$

$$Q_t \delta'(z_t) = r_t^k \tag{3.72}$$

$$Q_t = \beta \mathbb{E}_t \left[\frac{\widehat{\lambda}_{t+1}}{\widehat{\lambda}_t} \left(\frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}}\right)^{-\sigma_c} \Big(Q_{t+1}\big(1 - \delta(z_{t+1})\big) + r_{t+1}^k z_{t+1}\Big)\right] \tag{3.73}$$

Notice that Tobin's Q Q_t is a stationary variable. Therefore, by its definition, the Lagrange multiplier μ_t is trended. We write $\widehat{\mu}_t = \mu_t \mathcal{A}_{T,t}^{\sigma_c}$.

Let $K_t \equiv \widehat{K}_t \mathcal{A}_{T,t}$, and the dynamics for the stock of capital is

$$\widehat{K}_t = (1 - \delta(z_t)) \frac{\mathcal{A}_{T,t-1}}{\mathcal{A}_{T,t}} \widehat{K}_{t-1} + \varepsilon_{I,t} \left(1 - \mathcal{S}\left(\frac{\widehat{I}_t}{\widehat{I}_{t-1}} \frac{\mathcal{A}_{T,t}}{\mathcal{A}_{T,t-1}}\right)\right) \widehat{I}_t \tag{3.74}$$

On the production side, the factor price frontier becomes

$$\frac{\widehat{w}_t L_t^d}{r_t^k \widehat{K}_t^d} = \frac{1-\alpha}{\alpha}. \tag{3.75}$$

Marginal cost is written

$$mc_t = A_{C,t}^{\alpha-1}\left(\frac{r_t^k}{\alpha}\right)^{\alpha}\left(\frac{\widehat{w}_t}{1-\alpha}\right)^{1-\alpha} \tag{3.76}$$

For the price equations, we define $\mathscr{Z}_{i,t} = \widehat{\mathscr{Z}_{i,t}}\mathcal{A}_{T,t}^{1-\sigma_c}$ for $i = 1, 2$ and 3, then,

$$\frac{P_t^\star}{P_t} = \frac{\theta_f(1+\psi_f)}{\theta_f(1+\psi_f)-1}\frac{\widehat{\mathscr{Z}_{1,t}}}{\widehat{\mathscr{Z}_{2,t}}} + \frac{\psi_f}{\theta_f(1+\psi_f)-1}\left(\frac{P_t^\star}{P_t}\right)^{1+(1+\psi_f)\theta_f}\frac{\widehat{\mathscr{Z}_{3,t}}}{\widehat{\mathscr{Z}_{2,t}}} \tag{3.77}$$

and

$$\widehat{\mathscr{Z}_{1,t}} = \widehat{\lambda}_t mc_t \Theta_t^{(1+\psi_f)\theta_f}\widehat{Y}_t + \beta\xi_p\mathbb{E}_t\left[\left(\frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}}\right)^{1-\sigma_c}\left(\frac{\pi_{t+1}}{\pi_{t-1}^{\gamma_p}\bar{\pi}_t^{1-\gamma_p}}\right)^{(1+\psi_f)\theta_f}\widehat{\mathscr{Z}_{1,t+1}}\right] \tag{3.78a}$$

$$\widehat{\mathscr{Z}_{2,t}} = \widehat{\lambda}_t \varepsilon_{y,t}\Theta_t^{(1+\psi_f)\theta_f}\widehat{Y}_t + \beta\xi_p\mathbb{E}_t\left[\left(\frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}}\right)^{1-\sigma_c}\left(\frac{\pi_{t+1}}{\pi_{t-1}^{\gamma_p}\bar{\pi}_t^{1-\gamma_p}}\right)^{(1+\psi_f)\theta_f-1}\widehat{\mathscr{Z}_{2,t+1}}\right] \tag{3.78b}$$

$$\widehat{\mathscr{Z}_{3,t}} = \widehat{\lambda}_t \varepsilon_{y,t}\widehat{Y}_t + \beta\xi_p\mathbb{E}_t\left[\left(\frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}}\right)^{1-\sigma_c}\left(\frac{\pi_{t-1}^{\gamma_p}\bar{\pi}_t^{1-\gamma_p}}{\pi_{t+1}}\right)\widehat{\mathscr{Z}_{3,t+1}}\right] \tag{3.78c}$$

Equations (3.34) and (3.35) are unchanged.

For the unions and the employment agency, we define $\mathscr{H}_{i,t} = \widehat{\mathscr{H}_{i,t}}\mathcal{A}_{T,t}^{1-\sigma_c}$ for $i = 1, 2$ and 3, $w_t^\star = \mathcal{A}_{T,t}\widehat{w}_t^\star$ and $\varpi_t = \widehat{\varpi}_t\mathcal{A}_{T,t}/\mathcal{A}_{T,t-1}$. The relative wage of a union that has the possibility to reoptimize in period t is

$$\frac{\widehat{w}_t^\star}{\widehat{w}_t} = \frac{\theta_s(1+\psi_s)}{\theta_s(1+\psi_s)-1}\frac{\widehat{\mathscr{H}_{1,t}}}{\widehat{\mathscr{H}_{2,t}}} + \frac{\psi_s}{\theta_s(1+\psi_s)-1}\left(\frac{\widehat{w}_t^\star}{\widehat{w}_t}\right)^{1+(1+\psi_s)\theta_s}\frac{\widehat{\mathscr{H}_{3,t}}}{\widehat{\mathscr{H}_{2,t}}} \tag{3.79}$$

$$\widehat{\mathscr{H}_{1,t}} = \widehat{\lambda}_t\widehat{w}_t^m\mathcal{L}_t\Upsilon_t^{(1+\psi_s)\theta_s} + \beta\xi_w\mathbb{E}_t\left[\left(\frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}}\right)^{1-\sigma_c}\left(\frac{\widehat{\varpi}_{t+1}\pi_{t+1}}{\pi_t^{\gamma_w}\bar{\pi}_t^{1-\gamma_w}}\right)^{(1+\psi_s)\theta_s}\widehat{\mathscr{H}_{1,t+1}}\right] \tag{3.80a}$$

$$\widehat{\mathscr{H}_{2,t}} = \widehat{\lambda}_t \varepsilon_{l,t} \widehat{w}_t \mathcal{L}_t \Upsilon_t^{(1+\psi_s)\theta_s} + \beta \xi_w \mathbb{E}_t \left[\left(\frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}} \right)^{1-\sigma_c} \left(\frac{\widehat{\varpi}_{t+1} \pi_{t+1}}{\pi_t^{\gamma_w} \bar{\pi}_t^{1-\gamma_w}} \right)^{(1+\psi_s)\theta_s - 1} \widehat{\mathscr{H}_{2,t+1}} \right] \tag{3.80b}$$

$$\widehat{\mathscr{H}_{3,t}} = \widehat{\lambda}_t \varepsilon_{l,t} \widehat{w}_t \mathcal{L}_t + \beta \xi_w \mathbb{E}_t \left[\left(\frac{\mathcal{A}_{T,t+1}}{\mathcal{A}_{T,t}} \right)^{1-\sigma_c} \left(\frac{\pi_t^{\gamma_w} \bar{\pi}_t^{1-\gamma_w}}{\widehat{\varpi}_{t+1} \pi_{t+1}} \right) \widehat{\mathscr{H}_{3,t+1}} \right] \tag{3.80c}$$

$\vartheta_{s,t}$ can be expressed as a function of stationarized wages:

$$\vartheta_{s,t} = (1 - \xi_w) \frac{\widehat{w}_t^\star}{\widehat{w}_t} + \xi_w \frac{\bar{\pi}_t^{\gamma_w} \pi_{t-1}^{1-\gamma_w}}{\widehat{\varpi}_t \pi_t} \vartheta_{s,t-1} \tag{3.81}$$

Equation (3.53) remains unchanged. Finally, wage distortion is expressed as a function of stationarized wages:

$$\nabla_{w,t} = (1 - \xi_w) \left(\frac{\widehat{w}_t^\star}{\widehat{w}_t} \right)^{-(1+\psi_s)\theta_s} + \xi_w \left(\frac{\bar{\pi}_t^{\gamma_w} \pi_{t-1}^{1-\gamma_w}}{\widehat{\varpi}_t \pi_t} \right)^{-(1+\psi_s)\theta_s} \nabla_{w,t-1} \tag{3.82}$$

Equilibrium conditions on the good and factor markets are the same for the stationarized variables.

3.6.2 Stationary state

At the stationary state, we have

$$\left(\widehat{C} - \frac{\eta}{1+g} \widehat{C} \right)^{-\sigma_c} \exp\left\{ \widetilde{L} \frac{\sigma_c - 1}{1 + \sigma_l} \right\} = \widehat{\lambda} \tag{3.83}$$

$$\widehat{\lambda} = \beta R (1+g)^{-\sigma_c} \frac{\widehat{\lambda}}{\pi^\star} \tag{3.84}$$

$$\left[\widehat{C} - \frac{\eta}{1+g} \widehat{C} \right]^{1-\sigma_c} \frac{\widetilde{L}}{L} \exp\left\{ \widetilde{L} \frac{\sigma_c - 1}{1 + \sigma_l} \right\} = \widehat{\lambda} \widehat{w}^m \tag{3.85}$$

$$Q = 1 \tag{3.86}$$

$$\delta'\left(\bar{z}\right) = r^k \tag{3.87}$$

$$1 = \beta(1+g)^{-\sigma_c}\left(1 - \bar{\delta} + r^k \bar{z}\right) \tag{3.88}$$

$$\widehat{K} = \frac{1+g}{g+\bar{\delta}}\widehat{I} \tag{3.89}$$

$$\frac{\widehat{w}L^d}{r^k \widehat{K}^d} = \frac{1-\alpha}{\alpha} \tag{3.90}$$

$$mc = \left(\frac{r^k}{\alpha}\right)^{\alpha}\left(\frac{\widehat{w}}{1-\alpha}\right)^{1-\alpha} \tag{3.91}$$

$$\frac{P^\star}{P} = \frac{\theta_f(1+\psi_f)}{\theta_f(1+\psi_f)-1}\frac{\widehat{\mathscr{Z}_1}}{\widehat{\mathscr{Z}_2}} + \frac{\psi_f}{\theta_f(1+\psi_f)-1}\left(\frac{P^\star}{P}\right)^{1+(1+\psi_f)\theta_f}\frac{\widehat{\mathscr{Z}_3}}{\widehat{\mathscr{Z}_2}} \tag{3.92}$$

$$\widehat{\mathscr{Z}_1} = \widehat{\lambda} mc \Theta^{(1+\psi_f)\theta_f}\widehat{Y} + \beta\xi_p(1+g)^{1-\sigma_c}\widehat{\mathscr{Z}_1} \tag{3.93a}$$

$$\widehat{\mathscr{Z}_2} = \widehat{\lambda}\Theta^{(1+\psi_f)\theta_f}\widehat{Y} + \beta\xi_p(1+g)^{1-\sigma_c}\widehat{\mathscr{Z}_2} \tag{3.93b}$$

$$\widehat{\mathscr{Z}_3} = \widehat{\lambda}\widehat{Y} + \beta\xi_p(1+g)^{1-\sigma_c}\widehat{\mathscr{Z}_3} \tag{3.93c}$$

$$\vartheta_f = \frac{P^\star}{P} \tag{3.94}$$

$$\frac{\psi_f \vartheta_f}{1+\psi_f} + \frac{\Theta}{1+\psi_f} = 1 \tag{3.95}$$

$$\Delta_p = \frac{1}{1+\psi_f} \Theta^{(1+\psi_f)\theta_f} \nabla_p + \frac{\psi_f}{1+\psi_f} \tag{3.96a}$$

$$\nabla_p = \left(\frac{P_t^\star}{P_t}\right)^{-(1+\psi_f)\theta_f} \tag{3.96b}$$

$$\Theta = \frac{P^\star}{P} \tag{3.97}$$

$$\frac{\widehat{w}^\star}{\widehat{w}} = \frac{\theta_s(1+\psi_s)}{\theta_s(1+\psi_s)-1} \frac{\widehat{\mathscr{H}_1}}{\widehat{\mathscr{H}_2}} + \frac{\psi_s}{\theta_s(1+\psi_s)-1} \left(\frac{\widehat{w}^\star}{\widehat{w}}\right)^{1+(1+\psi_s)\theta_s} \frac{\widehat{\mathscr{H}_3}}{\widehat{\mathscr{H}_2}} \tag{3.98}$$

$$\widehat{\mathscr{H}_1} = \widehat{\lambda} \widehat{w}^m \mathcal{L} \Upsilon^{(1+\psi_s)\theta_s} + \beta \xi_w (1+g)^{1-\sigma_c} \widehat{\mathscr{H}_1} \tag{3.99a}$$

$$\widehat{\mathscr{H}_2} = \widehat{\lambda} \widehat{w} \mathcal{L} \Upsilon^{(1+\psi_s)\theta_s} + \beta \xi_w (1+g)^{1-\sigma_c} \widehat{\mathscr{H}_2} \tag{3.99b}$$

$$\widehat{\mathscr{H}_3} = \widehat{\lambda} \widehat{w} \mathcal{L} + \beta \xi_w (1+g)^{1-\sigma_c} \widehat{\mathscr{H}_3} \tag{3.99c}$$

$$\vartheta_s = \frac{\widehat{w}^\star}{\widehat{w}} \tag{3.100}$$

$$\frac{\psi_s \vartheta_s}{1+\psi_s} + \frac{\Upsilon}{1+\psi_s} = 1 \tag{3.101}$$

$$\Delta_w = \frac{1}{1+\psi_s} \Upsilon^{(1+\psi_s)\theta_s} \nabla_w + \frac{\psi_s}{1+\psi_s} \tag{3.102a}$$

$$\nabla_w = \left(\frac{\widehat{w}^\star}{\widehat{w}}\right)^{-(1+\psi_s)\theta_s} \tag{3.102b}$$

$$\Upsilon = 1 \tag{3.103}$$

$$R = R^\star \tag{3.104}$$

$$\Delta_p \widehat{Y} = \left(\widehat{K}^d\right)^\alpha \left(L^d\right)^{1-\alpha} \tag{3.105}$$

$$L = \Delta_w \mathcal{L} \tag{3.106}$$

$$L^d = \Delta_w \mathcal{L} \tag{3.107}$$

$$\frac{\bar{z}}{1+g}\widehat{K} = \widehat{K}^d \tag{3.108}$$

$$\widehat{C} + \widehat{I} = (1 - g^\star)\,\widehat{Y} \tag{3.109}$$

We can write equations (3.93a) to (3.93c) equivalently as

$$\widehat{\mathscr{X}_1} = \frac{\widehat{\lambda} mc \Theta^{(1+\psi_f)\theta_f}\widehat{Y}}{1 - \beta\xi_p(1+g)^{1-\sigma_c}} \tag{3.110a}$$

$$\widehat{\mathscr{X}_2} = \frac{\widehat{\lambda}\Theta^{(1+\psi_f)\theta_f}\widehat{Y}}{1 - \beta\xi_p(1+g)^{1-\sigma_c}} \tag{3.110b}$$

$$\widehat{\mathscr{X}_3} = \frac{\widehat{\lambda}\widehat{Y}}{1 - \beta\xi_p(1+g)^{1-\sigma_c}} \tag{3.110c}$$

Substituting in (3.92), we get

$$\frac{P^\star}{P} = \frac{\theta_f(1+\psi_f)}{\theta_f(1+\psi_f)-1}mc + \frac{\psi_f}{\theta_f(1+\psi_f)-1}\left(\frac{P^\star}{P}\right)^{1+(1+\psi_f)\theta_f}\Theta^{-(1+\psi_f)\theta_f} \tag{3.111}$$

that implicitly define real marginal cost at the stationary state as a function of optimal relative price ${P^\star}/{P}$ and the Lagrange multiplier from the optimization program for the firm that produces the homogeneous good. Furthermore, sustitution equation (3.97) in (3.95), (3.94), (3.96b) and (3.96a) we get

$$\Theta = \vartheta_f = \frac{P^\star}{P} = \nabla_p = \Delta_p = 1 \tag{3.112}$$

Marginal cost at the stationary state is then

$$mc = \frac{\theta_f(1+\psi_f)-1}{\theta_f(1+\psi_f)} - \frac{\psi_f}{\theta_f(1+\psi_f)} = \frac{\theta_f - 1}{\theta_f} \tag{3.113}$$

Finally, substituting (3.97) in equations (3.110a) to (3.110c) we obtain

$$\widehat{\mathscr{Z}_1} = mc\frac{\widehat{\lambda}\widehat{Y}}{1-\beta\xi_p(1+g)^{1-\sigma_c}} \tag{3.114a}$$

$$\widehat{\mathscr{Z}_2} = \widehat{\mathscr{Z}_3} = \widehat{\mathscr{Z}_1}/mc \tag{3.114b}$$

where $\widehat{\lambda}$, $\widehat{Y}$ are determined below iwth the other real variables.

We can write equations (3.99a) and (3.99c) in the following manner:

$$\widehat{\mathscr{H}_1} = \frac{\widehat{\lambda}\widehat{w}^m\mathcal{L}\Upsilon^{(1+\psi_s)\theta_s}}{1-\beta\xi_w(1+g)^{1-\sigma_c}} \tag{3.115a}$$

$$\widehat{\mathscr{H}_2} = \frac{\widehat{\lambda}\widehat{w}\mathcal{L}\Upsilon^{(1+\psi_s)\theta_s}}{1-\beta\xi_w(1+g)^{1-\sigma_c}} \tag{3.115b}$$

$$\widehat{\mathscr{H}_3} = \frac{\widehat{\lambda}\widehat{w}\mathcal{L}}{1-\beta\xi_w(1+g)^{1-\sigma_c}} \tag{3.115c}$$

Substituting in (3.98) we get

$$\frac{\widehat{w}^\star}{\widehat{w}} = \frac{\theta_s(1+\psi_s)}{\theta_s(1+\psi_s)-1}\frac{\widehat{w}^m}{\widehat{w}} + \frac{\psi_s}{\theta_s(1+\psi_s)-1}\left(\frac{\widehat{w}^\star}{\widehat{w}}\right)^{1+(1+\psi_s)\theta_s}\Upsilon^{-(1+\psi_s)\theta_s} \tag{3.116}$$

that implicitly defines the ratio of real wage received by the household to the real wage paid to the employment agency as a function of optimal real relative wage, $\widehat{w}^\star/\widehat{w}$, and the Lagrange multiplier from the optimization program of the employment agency.

Substituting equation (3.103) in equations (3.101), (3.100), (3.102b) and (3.102a) we obtain

$$\Upsilon = \vartheta_s = \frac{\widehat{w}^\star}{\widehat{w}} = \nabla_w = \Delta_w = 1 \tag{3.117}$$

Then, we have

$$\frac{\widehat{w}^m}{\widehat{w}} = \frac{\theta_s(1+\psi_s)-1}{\theta_s(1+\psi_s)} - \frac{\psi_s}{\theta_s(1+\psi_s)} \tag{3.118}$$

Finally, subsituting (3.103) in (3.115a) to (3.115c) we get

$$\widehat{\mathscr{H}_1} = \widehat{w}^m\frac{\widehat{\lambda}\mathcal{L}}{1-\beta\xi_w(1+g)^{1-\sigma_c}} \tag{3.119a}$$

$$\widehat{\mathscr{H}_2} = \widehat{\mathscr{H}_3} = \widehat{\mathscr{H}_1}\frac{\widehat{w}}{\widehat{w}^m} \tag{3.119b}$$

Equations (3.104) and (3.84) imply

$$R^\star = \frac{\pi^\star}{\beta}(1+g)^{\sigma_c} \tag{3.120}$$

In practive, we are estimating (or calibrating) $R^\star$ and $\pi^\star$ then we deduce β conditionally to the current estimation of the growth rate along the balanced growth path, g. From equation (3.88) we get the return on capital at the steady state:

$$r^k = \frac{(1+g)^{\sigma_c} - \beta(1-\bar{\delta})}{\beta \bar{z}} \tag{3.121}$$

where $\bar{\delta}$ and $\bar{z}$ are estimated (or calibrated). Equation (3.87) establishes the following constraints on the derivative of the depreciation function at the steady state

$$\delta'(\bar{z}) = \frac{(1+g)^{\sigma_c} - \beta(1-\bar{\delta})}{\beta \bar{z}} \tag{3.122}$$

Conditionally to the long run level of real marginal cost and return on capital, we obtain steady state for the average real wage (relative to labor efficiency trend) using equation (3.91):

$$\widehat{w} = \left[mc \left(\frac{\alpha}{r^k} \right)^{\alpha} \right]^{\frac{1}{1-\alpha}} (1-\alpha) \tag{3.123}$$

Along the balanced growth path, real wage growth rate is the same as the growth rate of labor efficiency:

$$\varpi = 1 + g \tag{3.124}$$

Substituting (3.123) in (3.113) we obtain the real wage received by the household (still normalized by labor efficiency trend) at the steady state:

$$\widehat{w}^m = \left(\frac{\theta_s(1+\psi_s) - 1}{\theta_s(1+\psi_s)} - \frac{\psi_s}{\theta_s(1+\psi_s)} \right) \left[mc \left(\frac{\alpha}{r^k} \right)^{\alpha} \right]^{\frac{1}{1-\alpha}} (1-\alpha) \tag{3.125}$$

As price and wage distortion is zero at the steady state, we get

$$L^d = \mathcal{L} = L \tag{3.126}$$

and

$$\widehat{Y} = \widehat{y} = \left(\widehat{K}^d \right)^{\alpha} \left(L^d \right)^{1-\alpha} \tag{3.127}$$

In what follows, we write $\chi_{X,Y}$ the ratio of two variables X and Y. Equation (3.90) and (3.126) gives the capital labor ratio at the steady state:

$$\chi_{\widehat{K}^d,L} = \frac{\widehat{w}}{r^k}\frac{\alpha}{1-\alpha} \tag{3.128}$$

Using equations (3.126), (3.127) and (3.128) it follows that product per capita at the steady state is

$$\chi_{\widehat{Y},L} = \chi_{\widehat{K}^d,L}^{\alpha} \tag{3.129}$$

On writes the equilibrium on the final good market (3.109) as

$$\chi_{\widehat{C},\widehat{Y}} + \chi_{\widehat{I},\widehat{Y}} = 1 - g^{\star} \tag{3.130}$$

Using the equilibrium condition on the market for capital (3.108) as well as equations (3.128), (3.129) and (3.89), one gets the share of investment in output at the steady state:

$$\chi_{\widehat{I},\widehat{Y}} = \frac{g+\bar{\bar{\delta}}}{\bar{z}}\chi_{\widehat{K}^d,L}^{1-\alpha} \tag{3.131}$$

We can then determine $\chi_{\widehat{C},\widehat{Y}}$ by complementarity, using equation (3.130). Dividing (3.85) by (3.83) one gets

$$\left(\widehat{C} - \frac{\eta}{1+g}\widehat{C}\right)\frac{\widetilde{L}}{L} = \widehat{w}^m$$

Multiplying and dividing the left handside by $\widehat{Y}$, one gets

$$\chi_{\widehat{C},\widehat{Y}}\chi_{\widehat{Y},L}\left(1 - \frac{\eta}{1+g}\right)\widetilde{L} = \widehat{w}^m$$

and then

$$\widetilde{L} = \frac{\widehat{w}^m}{\chi_{\widehat{C},\widehat{Y}}\chi_{\widehat{Y},L}}\frac{1+g}{1+g-\eta} \tag{3.132}$$

Then,

$$\widehat{Y} = \chi_{\widehat{Y},L}L \tag{3.133}$$

$$\widehat{C} = \chi_{\widehat{C},\widehat{Y}} \widehat{Y} \tag{3.134}$$

$$\widehat{I} = \chi_{\widehat{I},\widehat{Y}} \widehat{Y} \tag{3.135}$$

Finally, the steady state of the Lagrange multiplier associated with the household budget constraint is obtained from equation (3.83):

$$\widehat{\lambda} = \left(\widehat{C} - \frac{\eta}{1+g} \widehat{C} \right)^{-\sigma_c} \exp\left\{ \widetilde{L} \frac{1-\sigma_c}{1+\sigma_l} \right\} \tag{3.136}$$

3.7 Specifying functional forms

3.7.1 The depreciation function

The rate of depreciation of physical capital depends on the utilization rate of capital. This function must verify $\delta(0) = 0$ and $\delta(z)' > 0$ for any $z \in [0,1]$ and we write $\delta(\bar{z}) = \bar{\delta}$. We use the following functional form:

$$\delta(z) = \bar{\delta} e^{\frac{\bar{\delta}'}{\bar{\delta}}(z-\bar{z})} \tag{3.137}$$

where $\bar{\delta}' \equiv \delta'(\bar{z})$ is the first derivative of the rate of depreciation evaluated at the steady state. This parameter is linked to parameters g, σ_c, β, $\bar{\delta}$ and $\bar{z}$ thru equation (3.122) :

$$\bar{\delta}' = \frac{(1+g)^{\sigma_c} - \beta(1-\bar{\delta})}{\beta \bar{z}} > 0$$

This parameter should therefore be estimated or calibrated. An alternative to calibrate (or estimate) $\bar{\delta}$ is to calibrate (or estimate) the ratio $\chi_{\widehat{I},\widehat{Y}}$. One can show that

$$\bar{\delta} = \frac{\frac{\theta_f \chi_{\widehat{I},\widehat{Y}}}{\alpha\beta(\theta_f - 1)}(1+g)^{\sigma_c} - \frac{\theta_f \chi_{\widehat{I},\widehat{Y}}}{\alpha(\theta_f - 1)} - g}{1 - \frac{\theta_f \chi_{\widehat{I},\widehat{Y}}}{\alpha(\theta_f - 1)}}$$

It is possible to examine the data in order to form a prior on $\chi_{\widehat{I},\widehat{Y}}$. It is a flexible way of exploiting information about the relation between production and investment in the long run.

3.7.2 The investment adjustment cost function

Adjusting its investment level is costly for the household. This cost is described by function $\mathcal{S}$. One assumes that the cost is zero along the balance growth path, $\mathcal{S}(1+g)=0$, that $\mathcal{S}(1)'=0$, and the cost is convex,t $\mathcal{S}''>0$. We choose the following specification form:

$$\mathcal{S}(x) \equiv \frac{\psi}{2}(x-(1+g))^2 \tag{3.138}$$

where $x \equiv {}^{I_t}/_{I_{t-1}}$ is the growth factor for investment and ψ a parameter, real and positive, measuring the size of adjusmtent costs. It needs to be calibrated or estimated. We can then express the stationarized adjustment cost function for investment as

$$\mathcal{S}\left(\frac{\widehat{I}_t}{\widehat{I}_{t-1}}\right) \equiv \frac{\psi(1+g)^2}{2}\left(\frac{\widehat{I}_t}{\widehat{I}_{t-1}}\frac{A_{T,t}}{A_{T,t-1}}\frac{1}{1+g}-1\right)^2$$

3.7.3 Share of labor income in the long run

Parameter α in the Cobb–Douglas production function used by the firms of the intermediary good sector is linked to the share of labor income in value added in the long run. We could exploit this information to form a prior on α, or, alternatively, estimate the share of labor income in the long run and deduce from it the value of this technological parameter, as, at the steady state, we have

$$\frac{wL}{Y} = \frac{\theta_f - 1}{\theta_f}(1-\alpha)$$

or, equivalently,

$$\alpha = 1 - \frac{\theta_f}{\theta_f - 1}\frac{wL}{Y}$$

4 Estimation

4.1 Data

The model is estimated with quarterly data on the period 1994–2008 (3rd quarter). We use the following observed variables: GDP, private consumption expenditures, private investment (sum of private growth capital formation in residential buildings and in plant and equipment), consumer price index. For the short term nominal interest rate we use the Bank of Japan target rate of unsecured overnight call rate.

4.2 Priors

The following parameters were calibrated: the steady value of the depreciation rate (δ) is set to 0.025, a value usual for quarterly model. The inflation target was set to 0.5%. Given that on the estimation sample, monetary policy was very often constrained by the zero lower bound for nominal interest rate, this parameter is hard to set and hard to interpret. It certainly can't be interpreted as the inflation rate desired by the monetary authority, but rather as a parameter convenient to describe the policy that effectively took place during this crisis period.

The priors used for estimating the other parameters are described in the first three columns of the results tables. The are set so as to reflect the domain of definition of the parameters and their values are often found in the literature or in similar models estimated on US or European data. Given the relatively short estimation sample, they need to be relatively informative.

4.3 Estimation results

Following on the arguments developed in the first section of the treatment of trends in DSGE models, we perform two estimation. The first one uses the rate of growth of GDP, private consumption, private investment and CPI inflation. In the second, we use instead the logarithm of the level of these variables. The resuls are given in Tables 1 and 2. The prior and posterior distribution of the parameters are represented in Figures 1 to 10.

Comparing the estimation results, one observes that estimating in the growth rate or in the level of the variables doesn't affect greatly the estimation results. For several parameters, the posterior distribution is very close to the prior. This indicates that the parameters are not identified (or that the prior is too tight). This is in part the consequence of a relatively short estimation sample. One could consider using additional observable variables, notably for the behavior of the labor market, but it could also lead to the reformulation of a more parsimonious model.

It appears that the data suggest an estimated value for the probability of not receiving a positive signal for a price change, ξ_p, is 0.37, noticeably lower than the prior mean of 0.72, and than the usual findings for Europe or the U.S. If confirmed, this result could suggest smaller nominal rigidities on the good markets in Japan.

The estimation of the monetary policy rule suggests that the inertia of the policy rule as measured by ρ_R is less important than expected. On the other hand, the reaction to the output gap, r_y, is more important. Given the fact that the zero lower bound for nominal interest rate is binding over a large part of the sample, one should be careful not to infer too much from these results about the behavior of the central bank.

Finally, writing the model with the log of interest rate instead of interest rate itself—in order to insure that the nominal interest rate remains non–negative—doesn't affect the estimation results, even for the parameters of the monetary policy rule. In fact, it appears that if log–linear approximation of nominal interest rate is able to mechanically constrain the model to generate non–negative levels of the nominal interest rate, it fails to diffuse the consequence of the zero lower bound to the other parts of the model.

5 Conclusion

In this paper, we discuss the issue of integrating trends in DSGE models. We show how it is possible to estimate trends and cycle components simultaneously, without detrending the data.

In order to illustrate this methodology, we estimate a DSGE model with standard features on Japanese data. We explore two different approaches. One estimates the model on the growth rate of nonstationary variables, the other estimates with the original variables in level. In the latter case, we need to use the diffuse Kalman filter. Examination of the estimation results shows very little differences between both approaches. If this equivalence results is confirmed in other studies, it would mean that there is not much gain to use the most complicated procedure with the diffuse Kalman filter.

In this paper, we also study the possibility to compute a linear approximation of the logarithm of nominal interest rate in order to satisfy the zero lower bound. However, this procedure fails

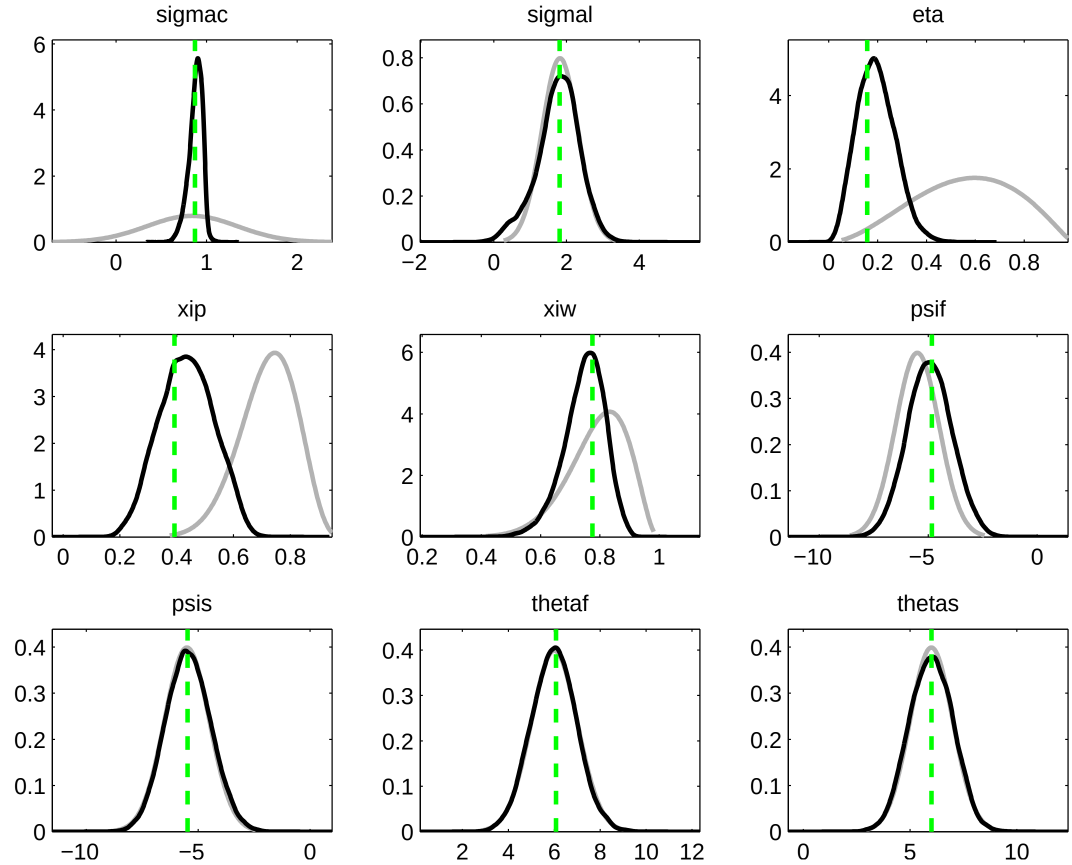

Figure 1: Estimation in growth rate: Prior and posterior distributions (I)

to transmit the effects of the zero lower bound to the rest of the model. This result confirms that there is no simple escape to the necessity of using nonlinear methods to handle the zero lower bound and these methods are not available for models of the size considered in this paper.

As often, the results in this paper are preliminary and should be completed by further study. An interesting development would be to estimate the same model on a longer dataset.

Parameter	Prior Distribution	Mean	S.D.	Posterior Mode	Mean	90% HPD interval	
param							
σ_c	Normal	0.840	0.5	0.8360	0.8514	0.7317	0.9776
σ_l	Normal	1.820	0.5	1.8374	1.7180	0.8957	2.5315
η	Beta	0.560	0.2	0.1544	0.1836	0.0615	0.2977
ξ_p	Beta	0.720	0.1	0.3700	0.4056	0.2759	0.5355
ξ_w	Beta	0.790	0.1	0.7891	0.7840	0.6965	0.8806
ψ_f	Normal	-5.500	1.0	-4.8195	-4.8380	-6.5855	-3.0893
ψ_s	Normal	-5.500	1.0	-5.4679	-5.5056	-7.1343	-3.8966
θ_f	Normal	6.000	1.0	6.1215	6.1786	4.5482	7.7292
θ_s	Normal	6.000	1.0	6.0068	5.9342	4.2381	7.5375
α	Beta	0.333	0.1	0.2955	0.3030	0.2441	0.3650
γ_p	Beta	0.290	0.1	0.1878	0.2079	0.0830	0.3347
γ_w	Beta	0.300	0.1	0.2528	0.2757	0.1237	0.4251
ρ_R	Beta	0.830	0.1	0.5811	0.5807	0.4576	0.7055
r_π	Normal	1.640	0.5	1.3620	1.4290	0.8806	1.9603
r_y	Normal	0.140	0.5	1.6246	1.7418	1.2600	2.2245
g	Normal	0.003	0.001	0.0033	0.0031	0.0019	0.0043
ψ	Normal	0.200	0.05	0.2251	0.2361	0.1686	0.3031
$\bar{z}$	Normal	0.800	0.1	0.7995	0.8012	0.6384	0.9662
ρ_{A_T}	Beta	0.500	0.1	0.3828	0.3919	0.2586	0.5313
ρ_{A_C}	Beta	0.700	0.1	0.6114	0.6001	0.4234	0.7670
ρ_{ε_B}	Beta	0.700	0.1	0.9267	0.9090	0.8628	0.9556
ρ_{ε_L}	Beta	0.700	0.1	0.7232	0.7035	0.5453	0.8797
ρ_{p_I}	Beta	0.700	0.1	0.6394	0.6223	0.4523	0.8005
ρ_{ε_I}	Beta	0.700	0.1	0.6069	0.6289	0.4688	0.7918
ρ_{ε_y}	Beta	0.700	0.1	0.8295	0.8050	0.6906	0.9283
ρ_{ε_l}	Beta	0.700	0.1	0.7468	0.7400	0.5857	0.9113
ρ_{ε_R}	Beta	0.700	0.1	0.6464	0.6388	0.4873	0.8002
ρ_{ε_g}	Beta	0.700	0.1	0.7284	0.7135	0.5566	0.8891
SE_{A_T}	InvGam	0.008	0.002	0.0055	0.0058	0.0045	0.0070
SE_{A_C}	InvGam	0.008	∞	0.0154	0.0209	0.0071	0.0356
SE_{ε_B}	InvGam	0.010	∞	0.0044	0.0048	0.0037	0.0058
SE_{ε_L}	InvGam	0.010	∞	0.0046	0.0077	0.0024	0.0139
SE_{p_I}	InvGam	0.010	∞	0.0045	0.0055	0.0028	0.0080
ε_I	InvGam	0.010	∞	0.0062	0.0061	0.0034	0.0087
ε_y	InvGam	0.010	∞	0.0043	0.0042	0.0032	0.0053
ε_l	InvGam	0.010	∞	0.0035	0.0040	0.0024	0.0056
ε_R	InvGam	0.010	∞	0.0047	0.0049	0.0036	0.0061
ε_g	InvGam	0.010	∞	0.0042	0.0055	0.0025	0.0085

Table 1: Estimation results when variables observed in growth rate

Parameter	Prior Distribution	Mean	S.D.	Posterior Mode	Mean	90% HPD interval	
param							
σ_c	Normal	0.840	0.5	0.8416	0.8733	0.7490	0.9941
σ_l	Normal	1.820	0.5	1.8109	1.7364	0.9046	2.5482
η	Beta	0.560	0.2	0.1470	0.1703	0.0559	0.2828
ξ_p	Beta	0.720	0.1	0.3723	0.4238	0.2802	0.5639
ξ_w	Beta	0.790	0.1	0.7912	0.7937	0.7073	0.8852
ψ_f	Normal	-5.500	1.0	-4.8310	-4.9054	-6.7187	-3.1115
ψ_s	Normal	-5.500	1.0	-5.4719	-5.4792	-7.1547	-3.8521
θ_f	Normal	6.000	1.0	6.0785	6.1190	4.5867	7.6886
θ_s	Normal	6.000	1.0	6.0015	5.9678	4.3716	7.6729
α	Beta	0.333	0.1	0.2909	0.3008	0.2396	0.3635
γ_p	Beta	0.290	0.1	0.1869	0.2072	0.0762	0.3326
γ_w	Beta	0.300	0.1	0.2533	0.2738	0.1170	0.4187
ρ_R	Beta	0.830	0.1	0.5884	0.5848	0.4609	0.7074
r_π	Normal	1.640	0.5	1.3831	1.4258	0.8678	1.9658
r_y	Normal	0.140	0.5	1.6456	1.7311	1.2517	2.2057
g	Normal	0.003	0.001	0.0032	0.0031	0.0019	0.0042
ψ	Normal	0.200	0.05	0.2263	0.2375	0.1667	0.3055
$\bar{z}$	Normal	0.800	0.1	0.8002	0.7999	0.6377	0.9666
ρ_{A_T}	Beta	0.500	0.1	0.3821	0.3906	0.2566	0.5257
ρ_{A_C}	Beta	0.700	0.1	0.6152	0.5819	0.4041	0.7563
ρ_{ε_B}	Beta	0.700	0.1	0.9315	0.9155	0.8741	0.9572
ρ_{ε_L}	Beta	0.700	0.1	0.7232	0.7153	0.5518	0.8824
ρ_{p_I}	Beta	0.700	0.1	0.6452	0.6277	0.4603	0.7998
ρ_{ε_I}	Beta	0.700	0.1	0.6084	0.6331	0.4633	0.7970
ρ_{ε_y}	Beta	0.700	0.1	0.8497	0.8233	0.7142	0.9329
ρ_{ε_l}	Beta	0.700	0.1	0.7491	0.7469	0.5925	0.9135
ρ_{ε_R}	Beta	0.700	0.1	0.6483	0.6492	0.4989	0.7992
ρ_{ε_g}	Beta	0.700	0.1	0.7287	0.7188	0.5574	0.8878
SE_{A_T}	InvGam	0.008	0.002	0.0054	0.0057	0.0045	0.0070
SE_{A_C}	InvGam	0.008	∞	0.0153	0.0242	0.0068	0.0436
SE_{ε_B}	InvGam	0.010	∞	0.0043	0.0047	0.0037	0.0057
SE_{ε_L}	InvGam	0.010	∞	0.0046	0.0129	0.0021	0.0274
SE_{p_I}	InvGam	0.010	∞	0.0044	0.0056	0.0029	0.0082
ε_I	InvGam	0.010	∞	0.0063	0.0062	0.0034	0.0088
ε_y	InvGam	0.010	∞	0.0044	0.0043	0.0032	0.0053
ε_l	InvGam	0.010	∞	0.0034	0.0041	0.0024	0.0057
ε_R	InvGam	0.010	∞	0.0047	0.0048	0.0036	0.0061
ε_g	InvGam	0.010	∞	0.0042	0.0056	0.0025	0.0088

Table 2: Estimation results when variables observed in level

Parameter	Prior Distribution	Mean	S.D.	Posterior Mode	Mean	90% HPD interval	
param							
σ_c	Normal	0.840	0.5	0.8363	0.8666	0.7484	0.9841
σ_l	Normal	1.820	0.5	1.8375	1.6822	0.8894	2.5228
η	Beta	0.560	0.2	0.1543	0.1816	0.0633	0.3008
ξ_p	Beta	0.720	0.1	0.3700	0.4159	0.2712	0.5629
ξ_w	Beta	0.790	0.1	0.7890	0.7883	0.6947	0.8752
ψ_f	Normal	-5.500	1.0	-4.8115	-4.8997	-6.6244	-3.1827
ψ_s	Normal	-5.500	1.0	-5.4605	-5.5124	-7.1231	-3.8352
θ_f	Normal	6.000	1.0	6.1878	6.1482	4.6174	7.7365
θ_s	Normal	6.000	1.0	6.0094	5.9689	4.3131	7.6758
α	Beta	0.333	0.1	0.2952	0.3057	0.2453	0.3672
γ_p	Beta	0.290	0.1	0.1882	0.2080	0.0793	0.3357
γ_w	Beta	0.300	0.1	0.2531	0.2727	0.1200	0.4172
ρ_R	Beta	0.830	0.1	0.5819	0.5836	0.4506	0.7044
r_π	Normal	1.640	0.5	1.3667	1.4289	0.8599	1.9573
r_y	Normal	0.140	0.5	1.6321	1.7370	1.2569	2.2119
g	Normal	0.003	0.001	0.0033	0.0031	0.0019	0.0042
ψ	Normal	0.200	0.05	0.2249	0.2357	0.1667	0.3021
$\bar{z}$	Normal	0.800	0.1	0.8000	0.8011	0.6386	0.9618
ρ_{A_T}	Beta	0.500	0.1	0.3824	0.3925	0.2594	0.5273
ρ_{A_C}	Beta	0.700	0.1	0.6115	0.5884	0.4122	0.7677
ρ_{ε_B}	Beta	0.700	0.1	0.9267	0.9110	0.8660	0.9549
ρ_{ε_L}	Beta	0.700	0.1	0.7232	0.7051	0.5433	0.8786
ρ_{p_I}	Beta	0.700	0.1	0.6396	0.6228	0.4554	0.7973
ρ_{ε_I}	Beta	0.700	0.1	0.6075	0.6292	0.4673	0.7934
ρ_{ε_y}	Beta	0.700	0.1	0.8293	0.7993	0.6844	0.9286
ρ_{ε_l}	Beta	0.700	0.1	0.7478	0.7420	0.5832	0.9042
ρ_{ε_R}	Beta	0.700	0.1	0.6472	0.6400	0.4849	0.7960
ρ_{ε_g}	Beta	0.700	0.1	0.7285	0.7175	0.5523	0.8736
SE_{A_T}	InvGam	0.008	0.002	0.0055	0.0057	0.0045	0.0070
SE_{A_C}	InvGam	0.008	∞	0.0154	0.0234	0.0066	0.0429
SE_{ε_B}	InvGam	0.010	∞	0.0044	0.0048	0.0038	0.0057
SE_{ε_L}	InvGam	0.010	∞	0.0046	0.0138	0.0023	0.0387
SE_{p_I}	InvGam	0.010	∞	0.0045	0.0056	0.0029	0.0081
ε_I	InvGam	0.010	∞	0.0062	0.0061	0.0034	0.0088
ε_y	InvGam	0.010	∞	0.0043	0.0042	0.0031	0.0053
ε_l	InvGam	0.010	∞	0.0035	0.0041	0.0024	0.0058
ε_R	InvGam	0.010	∞	0.0047	0.0048	0.0036	0.0060
ε_g	InvGam	0.010	∞	0.0042	0.0054	0.0025	0.0083

Table 3: Estimation results for approximation around the log of interest rate

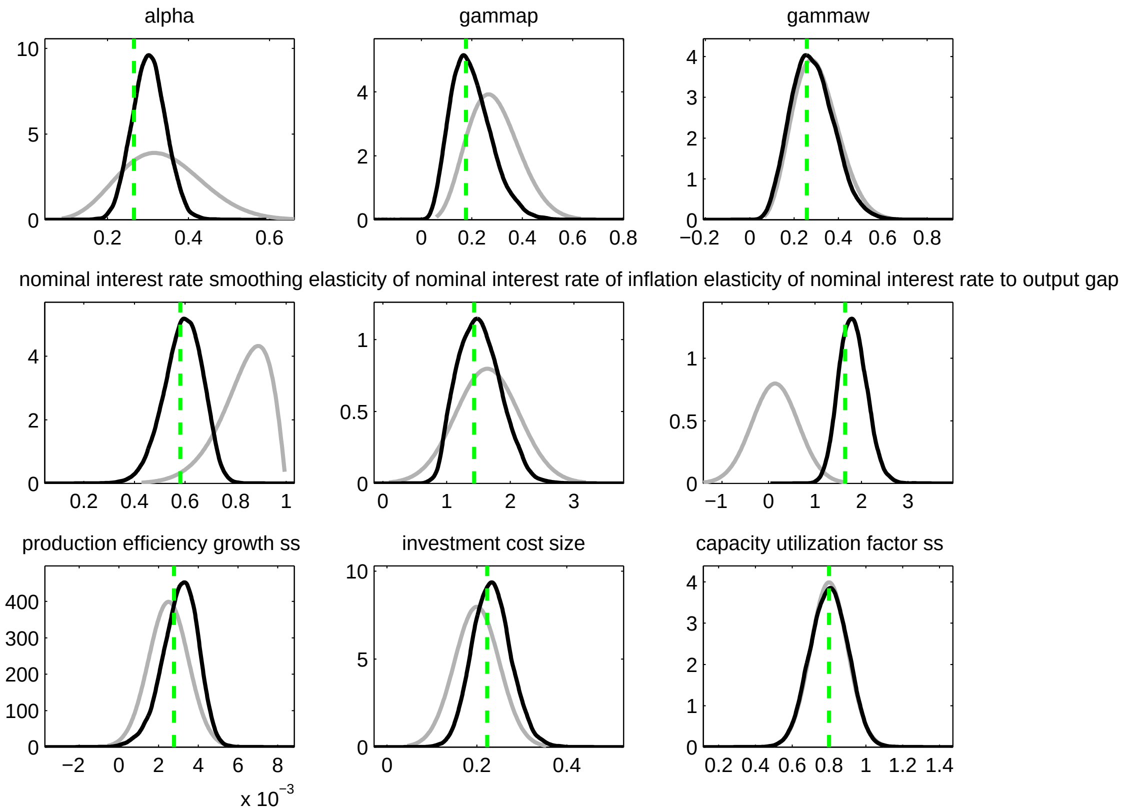

Figure 2: Estimation in growth rate: Prior and posterior distributions (II)

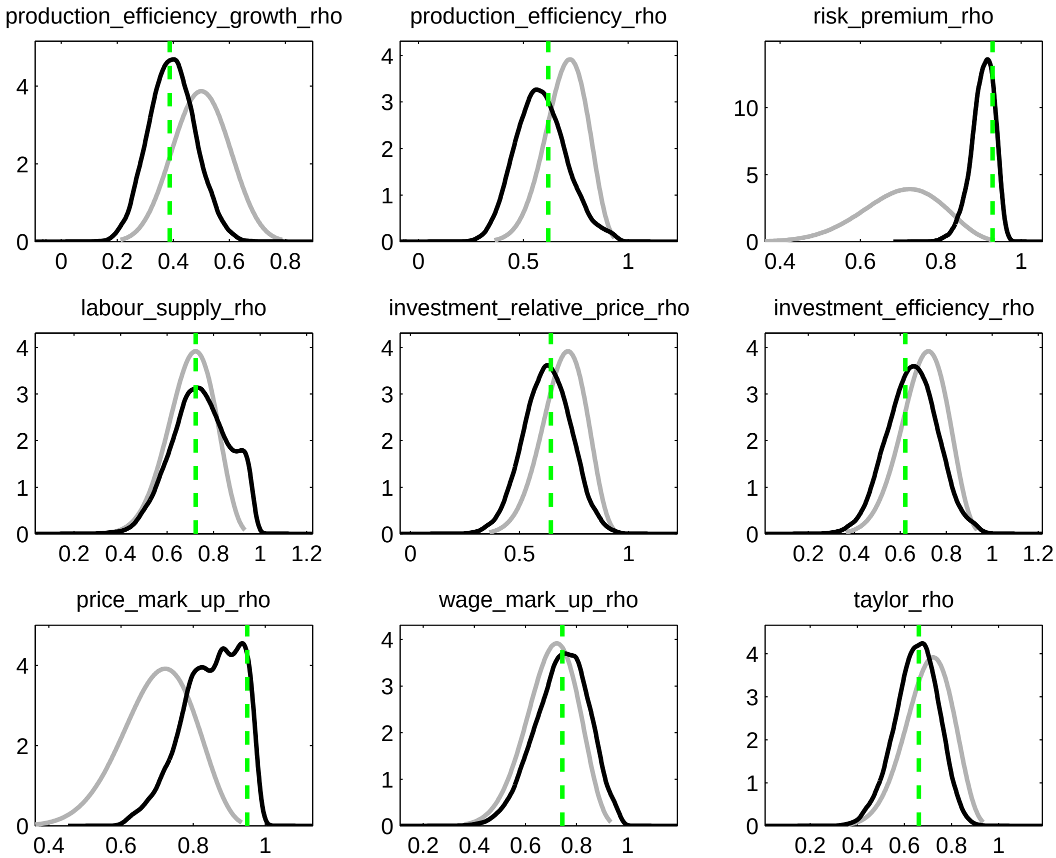

Figure 3: Estimation in growth rate: Prior and posterior distributions (III)

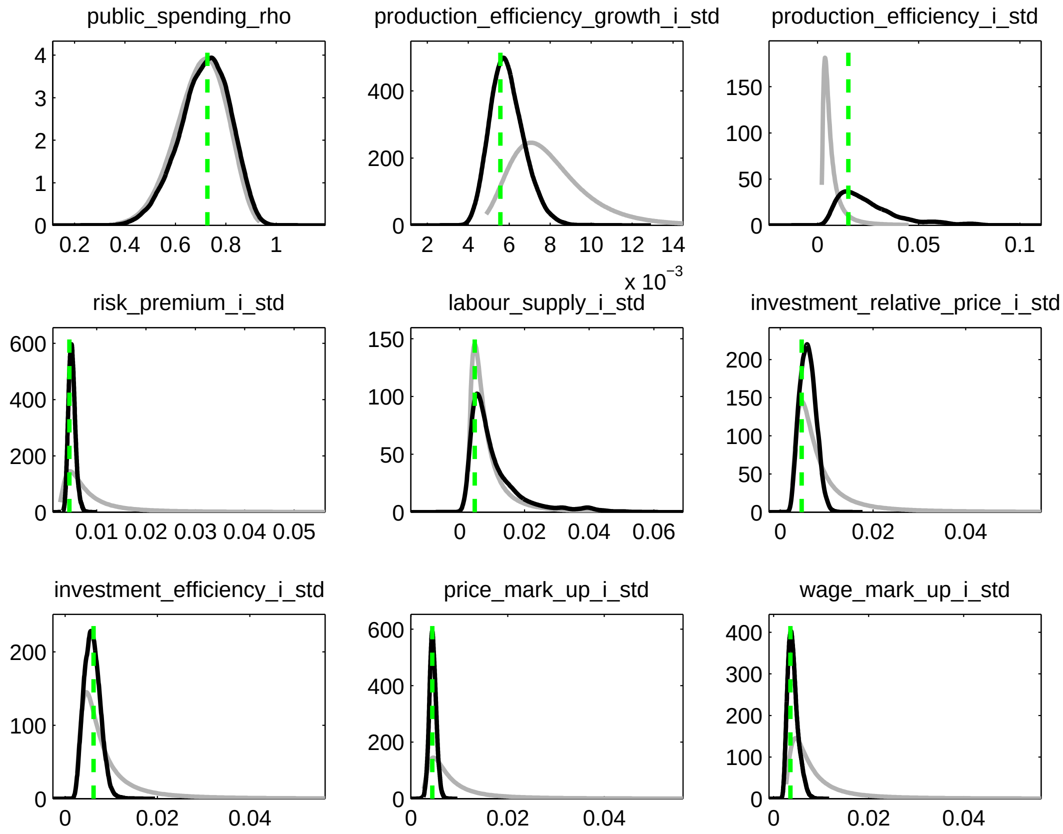

Figure 4: Estimation in growth rate: Prior and posterior distributions (IV)

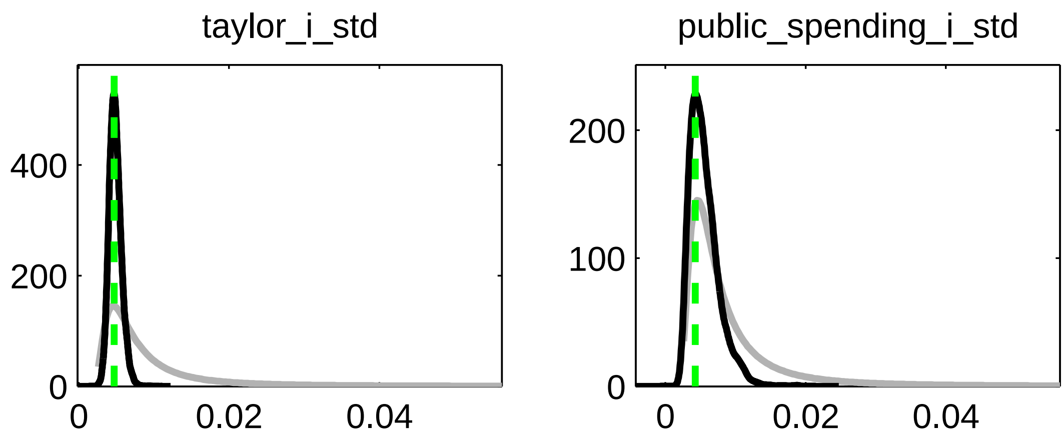

Figure 5: Estimation in growth rate: Prior and posterior distributions (V)

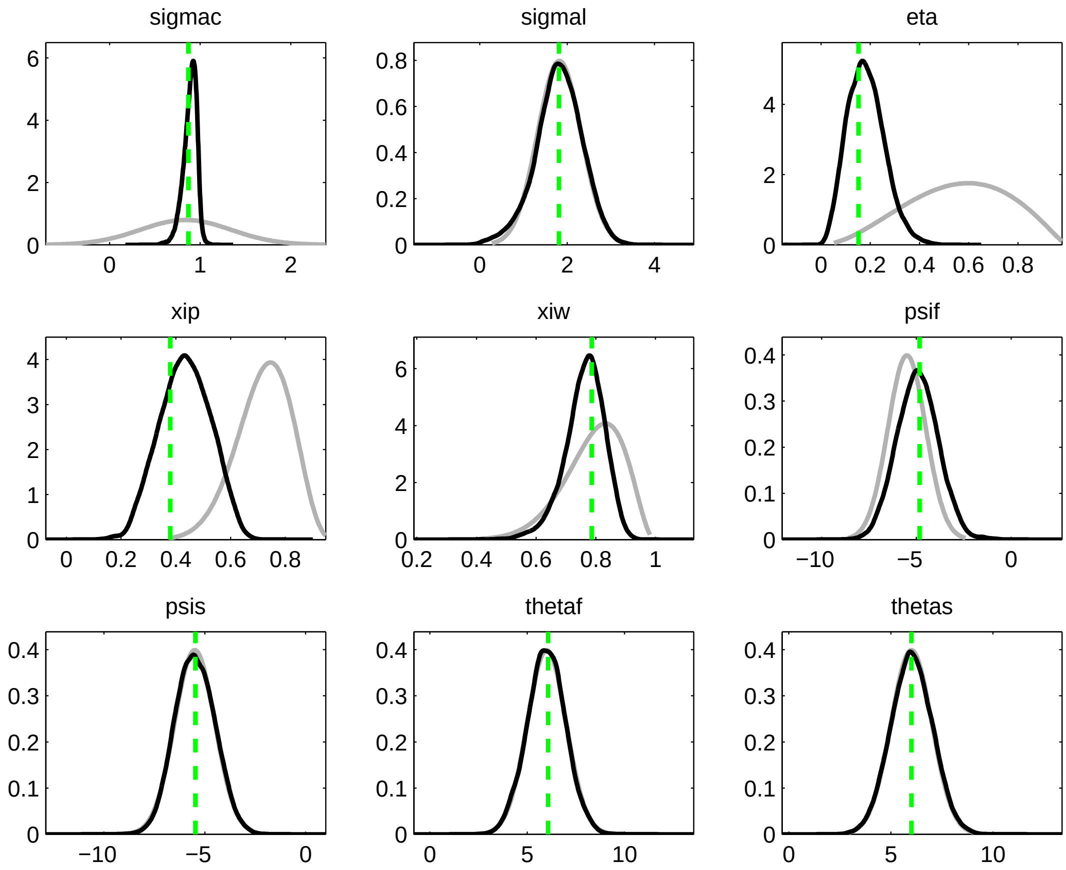

Figure 6: Estimation in level: Prior and posterior distributions (I)

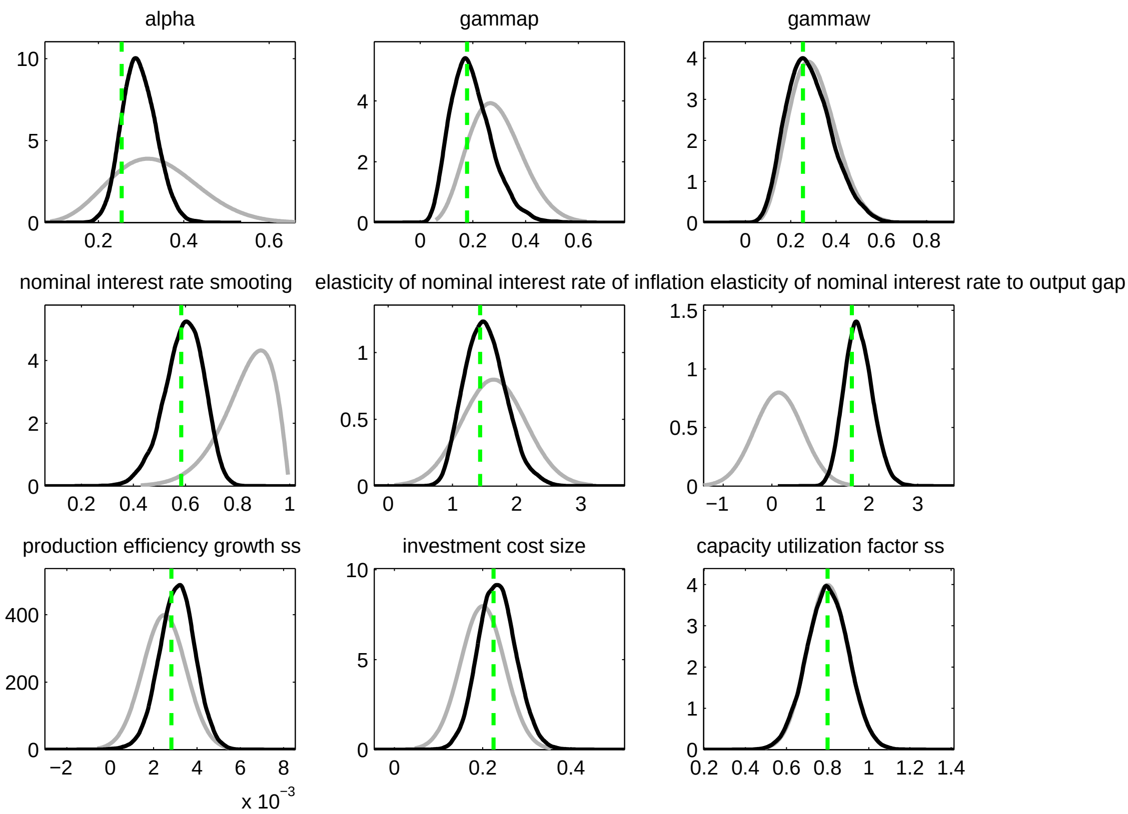

Figure 7: Estimation in level: Prior and posterior distributions (II)

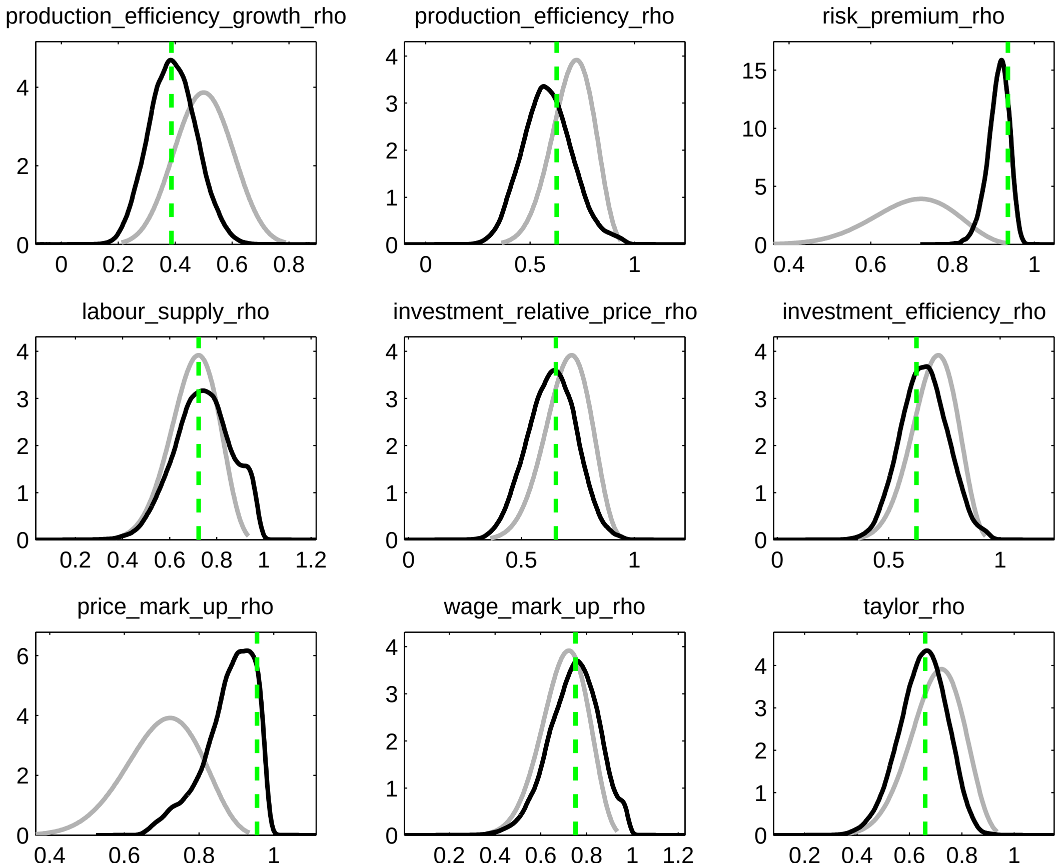

Figure 8: Estimation in level: Prior and posterior distributions (III)

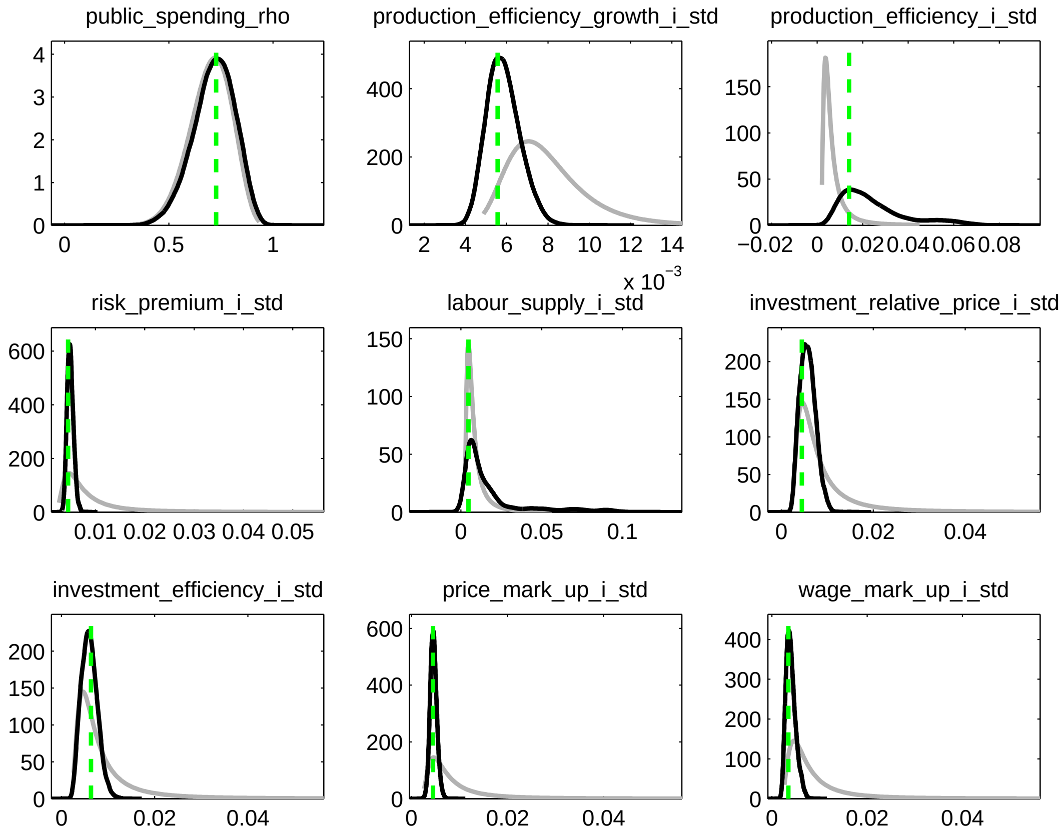

Figure 9: Estimation in level: Prior and posterior distributions (IV)

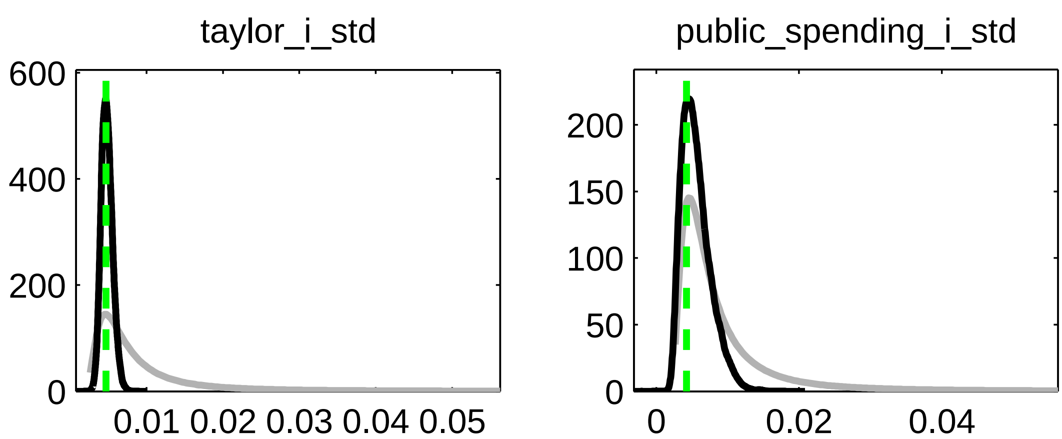

Figure 10: Estimation in level: Prior and posterior distributions (V)