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An application to the Japanese economy

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Abstract

In this paper we propose an estimation strategy for DSGE models with occasionaly binding constraints, such as models with a zero lower bound for the nominal interest rate (ZLB).

The usual likelihood approach is based on a first order approximation of the model around its deterministic steady state. This is not possible when we deal with a model with occasionally binding constraints, because the model is non differentiable everywhere and because, putting this first problem aside, the agents in the approximated model do not anticipate that the economy may hit the zero lower bound in the future.

A medium scaled DSGE model with ZLB is estimated by the Simulated Method of Moments, using the Extended Path approach to simulate artificial time series for the observed variables. The Extended Path approach to simulation of stochastic forward–looking models, takes into account the full nonlinearities of the deterministic part of the model, but ignores the Jensen inequality. The extended path method is well suited for models including the zero lower bound because (contrary to the perturbation method) it does not rely on a strong smoothness assumption and so can handle problems with non differentiabilities. This approach proves to be feasible in practice.

Introduction

Currently, estimation of DSGE models is generally done on the basis of a first order local approximation of the model. However, this approach is very unsatisfactory when the economy is confronted to an occasionally binding

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constraint such as the zero lower bound (ZLB) for nominal interest rates, because, with this solution strategy, the agents do not anticipate that the economy may hit the zero lower bound in the future.

A global approximation method would be the best alternative, but its computational burden would be very high for a medium-scaled model and such strategy is not applicable to this day.

In this paper, we try to estimate the model with the Simulated Method of Moments (SMM), using the Extended Path approach (EP) to simulate artificial time series for the observed variables. The Extended Path approach to simulation of stochastic forward–looking models, takes into account the full nonlinearities of the deterministic part of the model, but solves only approximatively the effect of future uncertainty by using the expected value of future shocks, zero by construction, instead of computing the expected nonlinear effects of future shocks, therefore ignoring Jensen inequality.

The extended path method is well suited for models including the zero lower bound because (contrary to the perturbation method) it does not rely on a strong smoothness assumption and so can handle problems with non differentiabilities. This approach proves to be feasible in practice.

In the case of the Japanese economy, the ZLB has been binding for almost ten years. It is a major challenge for the utilization of DSGE models in the case of Japan. The objective of this paper is to estimate a medium-scaled DSGE model for the Japanese economy over the recent period. In this paper, we evaluate the ability of a model, that we developed in a previous study (Adjemian and Juillard, 2009), to reproduce the sample frequency of hitting the ZLB, using GDP, consumption, investment, inflation, wages and nominal interest rate as observed variables.

In the first section we introduce briefly the model. The extended path method is presented in section 2 and the simulated method of moments in section 3. Section 4 presents first estimation results and concludes.

1 The model

The present model is strongly influenced by Smets and Wouters (2007). It is a closed economy model. Nominal rigidities *à la* Calvo on price and wage, with indexation on past inflation and steady state inflation, have a strong influence on the monetary transmission mechanism. So, do real rigidities on investment and endogenous capacity utilisation. The model contains a large number of shocks. We don't exploit all in the present paper.

1.1 Households

The economy is populated with a continuum of households $h \in [0, 1]$. Each household values consumption of a composite good. We write $C_t(h)$ the demand of this good by household h in period t. A household offers as well labor hours. We write $L_t(h)$ the labor supply of household h in period t. Welfare is defined as:

$$\begin{aligned} \mathcal{W}_t(h) &= u_t(h) + \beta \mathbb{E}[\mathcal{W}_{t+1}(h)] \\ u_t(h) &= \frac{\left[C_t(h) - \eta \overline{C}_{t-1}\right]^{1-\sigma_c}}{1-\sigma_c} \exp\left\{\varepsilon_{L,t} \frac{\sigma_c - 1}{1+\sigma_l} L_t(h)^{1+\sigma_l}\right\} \end{aligned} \tag{1.1}$$

where $\varepsilon_{L,t}$ is a shock to labor supply. $\log \varepsilon_{L,t}$ is an AR exogenous shock process with mean $\log \tilde{L}$ (this parameter gives us an extra degree of freedom for adjusting the stationary level of hours). We choose this form for the utility function in order to build a model compatible with balanced growth. We assume that utility obtained by household h in period t depends not only on its own consumption but as well on aggregate consumption in previous period, $\overline{C}_{t-1} = \int_0^1 C_{t-1}(h)\mathrm{d}h$. This is a mechanism of external habits.

The budget constraint of household h, in period t, in real terms, is the following:

$$\begin{aligned} C_t(h) + p_{I,t} I_t(h) = &\left\{ \frac{B_{t-1}(h)}{P_t} - \frac{B_t(h)}{P_t \varepsilon_{B,t} R_t} + (1-\tau_{W,t}) \frac{W_t^m}{P_t} L_t(h) \right. \\ &\left. + r_t^k z_t(h) K_{t-1}(h) + \frac{\mathscr{D}_{1,t}(h) + \mathscr{D}_{2,t}(h)}{P_t} \right\} + T_t \end{aligned} \tag{1.2}$$

where P_t is the aggregate price index; $R_t = 1 + i_t$, corresponds to the rate of interest plus one, $B_t(h)$ the nominal value of bonds detained by household h at the end of period t, ε_t^B is the risk premium requested by households

in order to detain the bond; $\log \varepsilon_t^B$ is an AR process with zero mean; $I_t(h)$ is investment of h during period t; $\log p_{I,t}$ is an exogenous shock on the relative price of investment and follows an AR process with zero mean; W_t^m is the hourly wage rate received by household h in period t; T_t represents net transfers received by the household during the period; $\mathscr{D}_{1,t}(h)$ and $\mathscr{D}_{2,t}(h)$ are the dividends received from firms and from the *unions* that differentiate household labor supply.
On the resource side, return on physical capital, r_t, is given by

$$r_t^k = z_t(h)K_{t-1}(h)$$

where the stock of physical capital at date t is

$$K_t(h) = (1 - \delta(z_t(h)))K_{t-1}(h) + \varepsilon_{I,t}\left(1 - \mathcal{S}\left(\frac{I_t(h)}{I_{t-1}(h)}\right)\right)I_t(h) \tag{1.3}$$

$z_t(h) \in [0,1]$ is the rate of utilization of physical capital with steady value $z^\star$; the depreciation rate, δ, is a function of the rate of utilization that verifies $\delta(0) = 0$, $\delta(1) = 1$, $\delta(z)' > 0$ for all $z \in [0,1]$ and we write $\delta(z^\star) = \delta^\star$; $\varepsilon_{I,t}$ is a random shock to the efficiency of capital accumulation $\log \varepsilon_{I,t}$ is an AR process with zero mean; function $\mathcal{S}$ describes adjustment costs on investment, we assume $\mathcal{S}(1+g) = 0$, where g is the rate of growth of investment on the balanced growth path, furthermore $\mathcal{S}(1+g)' = 0$ and $\mathcal{S}'' > 0$.

Each household h chooses its consumption, labor supply, bond holdings, investment, and capital utilization rate so as to maximize its inter-temporal utility (1.1) under the budget constraint (1.2) and the the law of evolution of physical capital (1.3), taking as given evolution of prices and exogenous variables.

The first order optimality conditions are given by:

$$\left(C_t(h) - \eta\overline{C}_{t-1}\right)^{-\sigma_c} \exp\left\{\varepsilon_{L,t}\frac{\sigma_c - 1}{1+\sigma_l}L_t(h)^{1+\sigma_l}\right\} = \lambda_t(h) \tag{1.4}$$

where $\lambda_t(h)$ is the Lagrange multiplier associated to the real budget constraint,

$$\lambda_t(h) = \beta\varepsilon_{B,t}R_t\mathbb{E}_t\left[\frac{\lambda_{t+1}(h)}{\pi_{t+1}}\right] \tag{1.5}$$

where $\pi_{t+1} \equiv P_{t+1}/P_t$ is the inflation rate between period t and $t+1$,

$$u_t(h)\left(\sigma_c - 1\right)\varepsilon_{L,t}L_t(h)^{\sigma_l} = -\lambda_t(h)\frac{W_t^m}{P_t}. \tag{1.6}$$

Writing $\mu_t(h)$ the Lagrange multiplier associated to the capital accumulation function one gets:

$$\begin{aligned} p_{I,t}\lambda_t(h) = \mu_t(h)\varepsilon_{I,t}\left[1 - \mathcal{S}\left(\frac{I_t(h)}{I_{t-1}(h)}\right) - \frac{I_t(h)}{I_{t-1}(h)}\mathcal{S}'\left(\frac{I_t(h)}{I_{t-1}(h)}\right)\right] \\ +\beta\mathbb{E}_t\left[\mu_{t+1}(h)\varepsilon_{I,t+1}\left(\frac{I_{t+1}(h)}{I_t(h)}\right)^2\mathcal{S}'\left(\frac{I_{t+1}(h)}{I_t(h)}\right)\right] \end{aligned} \tag{1.7}$$

$$\mu_t(h)\delta'\left(z_t(h)\right) = \lambda_t(h)r_t^k \tag{1.8}$$

and at last:

$$\mu_t(h) = \beta\mathbb{E}_t\Big[\mu_{t+1}(h)\Big(1 - \delta(z_{t+1}(h))\Big) + \lambda_{t+1}(h)r_{t+1}^k z_{t+1}(h)\Big] \tag{1.9}$$

Given the symmetrical nature of the solution for the household's problem, we get the following aggregated relationships:

$$\left(C_t - \eta C_{t-1}\right)^{-\sigma_c}\exp\left\{\varepsilon_{L,t}\frac{\sigma_c - 1}{1+\sigma_l}L_t^{1+\sigma_l}\right\} = \lambda_t \tag{1.10}$$

$$\lambda_t = \beta\varepsilon_{B,t}R_t\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\pi_{t+1}}\right] \tag{1.11}$$

$$u_t\varepsilon_{L,t}\left(\sigma_c - 1\right)L_t^{\sigma_l} = -\lambda_t\frac{W_t^m}{P_t} \tag{1.12}$$

$$\begin{aligned} \frac{p_{I,t}}{\varepsilon_{I,t}} =& Q_t\left[1 - \mathcal{S}\left(\frac{I_t}{I_{t-1}}\right) - \frac{I_t}{I_{t-1}}\mathcal{S}'\left(\frac{I_t}{I_{t-1}}\right)\right] \\ &+\beta\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}Q_{t+1}\frac{\varepsilon_{I,t+1}}{\varepsilon_{I,t}}\left(\frac{I_{t+1}}{I_t}\right)^2\mathcal{S}'\left(\frac{I_{t+1}}{I_t}\right)\right] \end{aligned} \tag{1.13}$$

$$Q_t\delta'\left(z_t\right) = r_t^k \tag{1.14}$$

$$Q_t = \beta\mathbb{E}_t\left[\frac{\lambda_{t+1}}{\lambda_t}\left(Q_{t+1}\Big(1 - \delta(z_{t+1})\Big) + r_{t+1}^k z_{t+1}\right)\right] \tag{1.15}$$

Here, $Q_t \equiv {}^{\mu_t}/_{\lambda_t}$ is Tobin's Q.

1.2 Production

1.2.1 Final good producers

Producers of final good, Y_t, operate in a perfectly competitive environment, assembling a continuum of diversified intermediary goods written $Y_t(\iota)$ with $\iota \in [0,1]$. They have access to a unique constant return aggregation technology as in Kimball (1996), implicitly defined by

$$\int_0^1 \mathscr{G}_f\left(\frac{Y_t(\iota)}{Y_t}\right) \mathrm{d}\iota = 1 \tag{1.16}$$

where $\mathscr{G}_f$ is a strictly increasing concave function such that $\mathscr{G}_f(1) = 1$. We follow Dotsey and King (2005) or Levin et al. (2007) and adopt the following functional form for this aggregation function:

$$\begin{aligned}\mathscr{G}_f(x) =& \frac{\theta_f(1+\psi_f)}{(1+\psi_f)(\theta_f(1+\psi_f)-1)}\left[(1+\psi_f)x-\psi_f\right]^{\frac{(1+\psi_f)\theta_f-1}{(1+\psi_f)\theta_f}} \\ &-\left[\frac{\theta_f(1+\psi_f)}{(1+\psi_f)(\theta_f(1+\psi_f)-1)}-1\right]\end{aligned} \tag{1.17}$$

Parameter ψ_f characterize the curvature of the demand function.

The producer of final good chooses the quantity of intermediary goods ι so as to maximize her real profit:

$$Y_t - \int_0^1 \frac{P_t(\iota)}{P_t} Y_t(\iota)\mathrm{d}\iota$$

under the technological constraints (1.16) and (1.17). As the aggregation function is homogeneous of degree one, it is equivalent to minimize cost per unit with respect to the relative demand of intermediary good ι under the technological constraint.

The first order condition of optimality determines the demand for intermediary good ι:

$$\frac{Y_t(\iota)}{Y_t} = \frac{1}{1+\psi_f}\left[\left(\frac{P_t(\iota)/P_t}{\Theta_t}\right)^{-(1+\psi_f)\theta_f} + \psi_f\right] \tag{1.18}$$

where Θ_t is the Lagrange multiplier associated with the technological constraints (1.16) and (1.17) for the representative firm. Substituting (1.18) in the technological constraints, one gets the following expression for the Lagrange multiplier:

$$\Theta_t = \left(\int_0^1 \left(\frac{P_t(\iota)}{P_t} \right)^{1-\theta_f(1+\psi_f)} \mathrm{d}\iota \right)^{\frac{1}{1-\theta_f(1+\psi_f)}} \tag{1.19}$$

The price elasticity of demand is given by:

$$\epsilon\left(\widetilde{Y}_t(\iota)\right) = -\frac{\mathscr{G}'\left(\widetilde{Y}_t(\iota)\right)}{\widetilde{Y}_t(\iota)\mathscr{G}''\left(\widetilde{Y}_t(\iota)\right)}$$

and, with particular aggregation function adopted in this study,

$$\epsilon\left(\widetilde{Y}_t(\iota)\right) = \theta_f \left[1 + \psi_f - \frac{\psi_f}{\widetilde{Y}_t(\iota)} \right]$$

When ψ_f is equal to zero, we get back to the more usual case of the CES aggregator of Dixit and Stiglitz (1977) with a price elasticity of demand equal to θ_f. More generally, one remarks that demand is more sensitive to price when the level of demand is important if and only if parameter ψ_f is positive. We expect therefore to obtain a negative value for this parameter.

Finally, as the final good sector is perfectly competitive, profit for the representative firm must be zero and we derive the aggregate price index:

$$P_t = \frac{\psi_f}{1+\psi_f} \int_0^1 P_t(\iota)\mathrm{d}\iota + \frac{1}{1+\psi_f} \left(\int_0^1 P_t(\iota)^{1-(1+\psi_f)\theta_f} \mathrm{d}\iota \right)^{\frac{1}{1-(1+\psi_f)\theta_f}} \tag{1.20}$$

1.2.2 Intermediary goods producers

A continuum of firms $\iota \in [0, 1]$ in monopolistic competition produce intermediary goods for the producers of the final good. These firms have all access to the same Cobb–Douglas technology in to transform physical capital and labor in differentiated intermediary goods:

$$Y_t(\iota) = \left(K_t^d(\iota)\right)^{\alpha} \left(A_t L_t^d(\iota)\right)^{1-\alpha} \tag{1.21}$$

where $K_t^d(\iota)$ and $L_t^d(\iota)$ are demands of intermediary good firm ι for physical capital, and labor, respectively; A_t is technical progress, neutral in Harrod sense. The latter term is further decomposed in a trend component $\mathcal{A}_{T,t}$ and a cyclical on $\mathcal{A}_{C,t}$. We have then,

$$\Delta \log \mathcal{A}_{T,t} \sim \text{AR}(1) \text{ stationary with mean } \log(1+g) \tag{1.22a}$$

$$\log \mathcal{A}_{C,t} \sim \text{AR}(1) \text{ stationary with zero mean.} \tag{1.22b}$$

Each intermediary firm $\iota \in [0,1]$ buys freely its production factors on competitive markets taking their price as given. The firm $\iota \in [0,1]$ decides upon the mix of physical capital $(K_t^d(\iota))$ and labor $(L_t^d(\iota))$ so as to minimize its cost, $r_t^k K_t^d(\iota) + w_t L_t^d(\iota)$, under the technological constraint (1.21). The firm optimal behavior on the factor markets is summarized by the following factor prices frontier:

$$\frac{w_t L_t^d(\iota)}{r_t^k K_t^d(\iota)} = \frac{1-\alpha}{\alpha} \tag{1.23}$$

where $w_t \equiv W_t/P_t$ is the real wage. The ratio of capital to labor is invariant across firms. Using the factor prices frontier, we rewrite the total cost of firm ι as a function of the stock of capital:

$$CT_t(\iota) = \frac{r_t^k K_t^d(\iota)}{\alpha}$$

On the other hand, as the returns of scale is constant, we know that the total cost can also be written as

$$CT_t(\iota) = mc_t(\iota) Y_t(\iota)$$

where $mc_t(\iota)$ is the real marginal cost. We derive then the following expression for the marginal cost of firm ι:

$$mc_t(\iota) = A_t^{\alpha-1} \left(\frac{r_t^k}{\alpha}\right)^{\alpha} \left(\frac{w_t}{1-\alpha}\right)^{1-\alpha} \equiv mc_t \tag{1.24}$$

Again, marginal cost doesn't depend on the size of the firm and is constant across firms.

The nominal profit of a firm that offers price $\mathcal{P}$ at date t id given by:

$$\Pi_t(\mathcal{P}) = \left(\varepsilon_{y,t} \frac{\mathcal{P}}{P_t} - mc_t\right) \frac{P_t Y_t}{1+\psi_f} \times \left[\left(\frac{\mathcal{P}}{P_t}\right)^{-(1+\psi_f)\theta_f} \left(\int_0^1 \left(\frac{P_t(\iota)}{P_t}\right)^{1-(1+\psi_f)\theta_f} \mathrm{d}f \right)^{\frac{(1+\psi_f)\theta_f}{1-(1+\psi_f)\theta_f}} + \psi_f \right]$$

where $\log \varepsilon_{y,t}$, a zero mean AR stationary process, is a shock to the sales of the firm. This shock will show up in the Phillips curve.

Firm ι has market power but can't decide of the its optimal price in each period. Following a Calvo scheme, at each date, the firm receives a signal telling it whether it can revise its price $P_t(\iota)$ in an optimal manner or not. There is a probability ξ_p that the firm can't revise its price in a given period. In such a case, the firm follows the following rule:

$$P_t(\iota) = [\bar{\pi}_t]^{\gamma_p} \left[\frac{P_{t-1}}{P_{t-2}}\right]^{1-\gamma_p} P_{t-1}(\iota) = \Gamma_t P_{t-1}(\iota) \tag{1.25}$$

where $\bar{\pi}_t$ is the inflation target of the monetary authorities. More generally, we write

$$\Gamma_{t+j,t} = \left(\prod_{h=0}^{j-1} \bar{\pi}_{t+h}\right)^{\gamma_p} \left(\prod_{h=0}^{j-1} \pi_{t+h}\right)^{1-\gamma_p} = \Gamma_{t+1}\Gamma_{t+2}\dots\Gamma_{t+j}$$

the growth factor of the price of a firm that doesn't receive a favorable signal during j successive periods(for $j = 0$ we have $\Gamma_{t,t} = 1$; for $j = 1$, we have $\Gamma_{t+1,t} = \Gamma_{t+1}$). When the firm ι receives a positive signal (with probability $1 - \xi_p$), it chooses price $P_t(\iota)$ that maximizes its profit.

Let $\widetilde{\mathscr{V}}_t$ be the value of a firm that receives a positive signal in period t and $\mathscr{V}_t(P_{t-1}(\iota))$ the value of a firm that receives a negative signal. As a firm that receives a negative signal follows simply the *ad hoc* pricing rule (1.25), its value at time t depends only on $P_{t-1}(\iota)$. For a firm that receives a positive signal, its value at period t is

$$\widetilde{\mathcal{V}}_t = \max_{\mathbf{P}} \left\{ \Pi_t\Big(\mathbf{P}\Big) + \beta \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \Big((1-\xi_p)\widetilde{\mathcal{V}}_{t+1} + \xi_p \mathcal{V}_{t+1}\big(\mathbf{P}\big) \Big) \right] \right\} \tag{1.26}$$

where Λ_t is the Lagrange multiplier of the budget constraint of the representative household and $P_t\Lambda_t = \lambda_t$. Let $P^\star$ be the optimal price chosen by the firm that can re–optimize. The value of a firm that can't re–optimize is

$$\begin{aligned} \mathcal{V}_t\big(P_{t-1}(\iota)\big) =& \Pi_t\Big(\Gamma_t P_{t-1}(\iota)\Big) \\ &+ \beta \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \Big((1-\xi_p)\widetilde{\mathcal{V}}_{t+1} + \xi_p \mathcal{V}_{t+1}\big(\Gamma_t P_{t-1}(\iota)\big) \Big) \right] \end{aligned} \tag{1.27}$$

The first order condition and the envelope theorem give

$$\Pi_t'\big(P^\star\big) + \beta \xi_p \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \mathcal{V}_{t+1}'\big(P^\star\big) \right] = 0 \tag{1.28a}$$

$$\frac{\mathcal{V}_t'\big(P_{t-1}(\iota)\big)}{\Gamma_t} = \Pi_t'\big(\Gamma_t P_{t-1}(\iota)\big) + \beta\xi_p \mathbb{E}_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\mathcal{V}_{t+1}'\big(\Gamma_t P_{t-1}(\iota)\big)\right] \tag{1.28b}$$

with the derivative of profit at $\mathcal{P}$:

$$\begin{aligned}\Pi_t'(\mathcal{P}) \quad &= \quad \varepsilon_{y,t}\frac{1-\theta_f(1+\psi_f)}{1+\psi_f}\left(\frac{\mathcal{P}}{P_t}\right)^{-(1+\psi_f)\theta_f}\Theta_t^{(1+\psi_f)\theta_f}Y_t \\ &\quad +\theta_f\left(\frac{\mathcal{P}}{P_t}\right)^{-(1+\psi_f)\theta_f-1}\Theta_t^{(1+\psi_f)\theta_f}mc_tY_t + \frac{\psi_f}{1+\psi_f}\varepsilon_{y,t}Y_t\end{aligned} \tag{1.29}$$

Let's write temporarily, in order to simplify notations, $\mathcal{P}$, the price inherited from the past. One can rewrite, one period ahead

$$\mathcal{V}_{t+1}'(\mathcal{P}) = \Gamma_{t+1,t}\Pi_{t+1}'\big(\Gamma_{t+1,t}\mathcal{P}\big) + \beta\xi_p\Gamma_{t+1,t}\mathbb{E}_{t+1}\left[\frac{\Lambda_{t+2}}{\Lambda_{t+1}}\mathcal{V}_{t+2}'\big(\Gamma_{t+1,t}\mathcal{P}\big)\right]$$

Iterating toward the future and applying conditional expectation at time t, one gets

$$\mathbb{E}_t\left[\mathcal{V}_{t+1}'(\mathcal{P})\right] = \mathbb{E}_t\left[\sum_{j=0}^{\infty}(\beta\xi_p)^j\Gamma_{t+1+j,t}\frac{\Lambda_{t+1+j}}{\Lambda_{t+1}}\Pi_{t+1+j}'\big(\Gamma_{t+1+j,t}\mathcal{P}\big)\right]$$

By substitution ($\mathcal{P} = P^\star$) in the first order condition, one gets the following condition for the price chosen by the firm that gets a positive signal:

$$\mathbb{E}_t\left[\sum_{j=0}^{\infty}(\beta\xi_p)^j\Gamma_{t+j,t}\frac{\Lambda_{t+j}}{\Lambda_t}\Pi_{t+j}'\big(\Gamma_{t+j,t}P_t^\star\big)\right] = 0 \tag{1.30}$$

One can get a more explicit expression for the price that satisfies equation (1.30). Substituting in this equation the expression of marginal profit (1.29) and dividing by $P_t^{\star\,-(1+\psi_f)\theta_f}$ one gets:

$$\frac{P_t^\star}{P_t} = \frac{\theta_f(1+\psi_f)}{\theta_f(1+\psi_f)-1}\frac{\mathscr{Z}_{1,t}}{\mathscr{Z}_{2,t}} + \frac{\psi_f}{\theta_f(1+\psi_f)-1}\left(\frac{P_t^\star}{P_t}\right)^{1+(1+\psi_f)\theta_f}\frac{\mathscr{Z}_{3,t}}{\mathscr{Z}_{2,t}} \tag{1.31}$$

with

$$\mathscr{Z}_{1,t} = \mathbb{E}_t\sum_{i=0}^{\infty}(\beta\xi_p)^j\lambda_{t+j}\left(\frac{\Gamma_{t+j}}{P_{t+j}/P_t}\right)^{-(1+\psi_f)\theta_f}\Theta_{t+j}^{(1+\psi_f)\theta_f}mc_{t+j}Y_{t+j} \tag{1.32a}$$

$$\mathscr{Z}_{2,t} = \mathbb{E}_t\sum_{i=0}^{\infty}(\beta\xi_p)^j\lambda_{t+j}\varepsilon_{y,t+j}\left(\frac{\Gamma_{t+j}}{P_{t+j}/P_t}\right)^{1-(1+\psi_f)\theta_f}\Theta_{t+j}^{(1+\psi_f)\theta_f}Y_{t+j} \tag{1.32b}$$

$$\mathscr{Z}_{3,t} = \mathbb{E}_t \sum_{i=0}^{\infty} (\beta\xi_p)^j \lambda_{t+j}\varepsilon_{y,t+j} \frac{\Gamma_{t+j}}{P_{t+j}/P_t} Y_{t+j} \tag{1.32c}$$

writing P_{t+j}/P_t, the inflation factor between t and $t+j$, can be written equivalently $\Pi_{i=1}^{j}\pi_{t+i}$, and we can represent variables $\mathscr{Z}_{1,t}$, $\mathscr{Z}_{2,t}$ et $\mathscr{Z}_{3,t}$ in recursive form:

$$\mathscr{Z}_{1,t} = \widehat{\lambda}_t mc_t \Theta_t^{(1+\psi_f)\theta_f} \widehat{Y}_t + \beta\xi_p \mathbb{E}_t \left[\left(\frac{\pi_{t+1}}{\pi_{t-1}^{\gamma_p} \bar{\pi}_t^{1-\gamma_p}} \right)^{(1+\psi_f)\theta_f} \mathscr{Z}_{1,t+1} \right] \tag{1.33a}$$

$$\mathscr{Z}_{2,t} = \widehat{\lambda}_t \varepsilon_{y,t} \Theta_t^{(1+\psi_f)\theta_f} \widehat{Y}_t + \beta\xi_p \mathbb{E}_t \left[\left(\frac{\pi_{t+1}}{\pi_{t-1}^{\gamma_p} \bar{\pi}_t^{1-\gamma_p}} \right)^{(1+\psi_f)\theta_f - 1} \mathscr{Z}_{2,t+1} \right] \tag{1.33b}$$

$$\mathscr{Z}_{3,t} = \widehat{\lambda}_t \varepsilon_{y,t} \widehat{Y}_t + \beta\xi_p \mathbb{E}_t \left[\left(\frac{\pi_{t-1}^{\gamma_p} \bar{\pi}_t^{1-\gamma_p}}{\pi_{t+1}} \right) \mathscr{Z}_{3,t+1} \right] \tag{1.33c}$$

Writing $\vartheta_{f,t} \equiv \int_0^1 \frac{P_t(\iota)}{P_t} \mathrm{d}\iota$ can be written in recursive form:

$$\vartheta_{f,t} = (1-\xi_p) \frac{P_t^\star}{P_t} + \xi_p \frac{\bar{\pi}_t^{1-\gamma_p} \pi_{t-1}^{\gamma_p}}{\pi_t} \vartheta_{f,t-1} \tag{1.34}$$

we can finally write the equation (1.20) equivalently as:

$$\frac{\psi_f \vartheta_{f,t}}{1+\psi_f} + \frac{\Theta_t}{1+\psi_f} = 1 \tag{1.35}$$

In the end, inflation dynamics are characterized by equations (1.35), (1.34), (1.31), (1.33a), (1.33b), (1.33c).

1.3 Labor

Homogeneous labor $L_t = \int_0^1 L_t(h)\mathrm{d}h$ provided by the households is differentiated by a continuum of unions, $\varsigma \in [0,1]$. We have then $L_t = \int_0^1 l_t(\varsigma)\mathrm{d}\varsigma$. Unions sell differentiated labor, $l_t(\varsigma)$, to an employment agency that aggregate different types of labor to offer it as input to the firms of the intermediary good sector. Unions have monopolistic power and the employment agency operates in a perfectly competitive manner.

1.3.1 Employment agency

It aggregates labor $l_t(\varsigma)$ provided by unions with an aggregation function as in Kimball (1996), defined implicitly by

$$\int_0^1 \mathscr{G}_s\left(\frac{l_t(\varsigma)}{\mathcal{L}_t}\right)\mathrm{d}\varsigma = 1 \tag{1.36}$$

where $\mathscr{G}_s$, a strictly increasing concave function such that $\mathscr{G}_s(1) = 1$, is defined as $\mathscr{G}_f$ in section 1.2 (replacing θ_f by θ_s and ψ_f by θ_s). The employment agency chooses the relative quantity of labor of type ς such as minimizing the cost of production by unit of homogeneous labor, $\frac{W_t(\varsigma)}{W_t}\frac{l_t(\varsigma)}{\mathcal{L}_t}$, under the technological constraint (1.36). The first order condition associated to the optimization program of the employment agency determines its demand of differentiated labor[1] ς:

$$\frac{l_t(\varsigma)}{\mathcal{L}_t} = \frac{1}{1+\psi_s}\left[\left(\frac{W_t(\varsigma)/W_t}{\Upsilon_t}\right)^{-(1+\psi_s)\theta_s} + \psi_s\right] \tag{1.37}$$

where Υ_t is the Lagrange multiplier associated with the technological constraint (1.36). Substituting (1.37) in the technological constraint, one gets the following expression for the Lagrange multiplier

$$\Upsilon_t = \left(\int_0^1 \left(\frac{W_t(\varsigma)}{W_t}\right)^{1-\theta_s(1+\psi_s)}\mathrm{d}\varsigma\right)^{\frac{1}{1-\theta_s(1+\psi_s)}} \tag{1.38}$$

As the employment agency behaves in a competitive manner, its profit is zero and we get the aggregate wage as

$$W_t = \frac{\psi_s}{1+\psi_s}\int_0^1 W_t(\varsigma)\mathrm{d}\varsigma + \frac{1}{1+\psi_s}\left(\int_0^1 W_t(\varsigma)^{1-(1+\psi_s)\theta_s}\mathrm{d}\varsigma\right)^{\frac{1}{1-(1+\psi_s)\theta_s}} \tag{1.39}$$

1.3.2 Unions

Unions supply differentiated labor services from the homogeneous labor supply from the households. Unions have market power because of this differentiation of labor services. We write the profit of a union offering wage $W_t(\varsigma)$ and $l_t(\varsigma)$ units of labor:

$$\mathscr{S}_t\left(W_t(\varsigma)\right)) = \left(\varepsilon_{l,t}W_t(\varsigma) - W_t^m\right)l_t(\varsigma)$$

[1] In order to save in notations, we don't make a difference between the demand of the employment agency and the supply by the unions.

where $\log \varepsilon_{l,t}$, is an exogenous shock on union's gains. It is an AR(1) process with zero mean. For a given demand, when $\varepsilon_{l,t} = 1$, the profit of the union is given by the difference between the wage asked to the employment agency and the wage paid to the household. By substitution in the demand function of the employment agency (1.37):

$$\mathscr{S}_t\left(W_t(\varsigma)\right) = \left(\varepsilon_{l,t} W_t(\varsigma) - W_t^m\right) \frac{\mathcal{L}_t}{1+\psi_s} \left[\left(\frac{W_t(\varsigma)/W_t}{\Upsilon_t}\right)^{-(1+\psi_s)\theta_s} + \psi_s\right] \tag{1.40}$$

Each union is subject to a Calvo lottery. In each period, a union can adjust the wage $W_t(\varsigma)$ in an optimal manner with probability ξ_w. In this case, the union chooses wage $W_t^\star$ that maximizes its profit knowing that in the future it may not have the opportunity to readjust the wage for some periods. When the lottery draw is negative for the union (with probability $1-\xi_w$), it adjusts wages according to the following *ad hoc* rule:

$$W_t(\varsigma) = \frac{\mathcal{A}_{T,t}}{\mathcal{A}_{T,t-1}} \bar{\pi}_{t-1}^{\gamma_w} \pi_{t-1}^{1-\gamma_w} W_{t-1}(\varsigma) \tag{1.41}$$

We write $\Omega_t = \left(\mathcal{A}_{T,t}/\mathcal{A}_{T,t-1}\right) \bar{\pi}_{t-1}^{\gamma_w} \pi_{t-1}^{1-\gamma_w} \equiv \Omega_{t,t-1}$ the growth factor of nominal wage asked by the union ς at date t when this one doesn't have the opportunity of revising it in an optimal manner. In this case, the union changes the wage by indexing it on (i) a convex mix of the inflation target of the monetary authority and of past inflation and (ii) the efficiency growth in the intermediary goods sector. We write

$$\Omega_{t+j,t} = \frac{\mathcal{A}_{T,t+j}}{\mathcal{A}_{T,t}} \left(\prod_{h=0}^{j-1} \bar{\pi}_{t+h}\right)^{\gamma_w} \left(\prod_{h=0}^{j-1} \pi_{t+h}\right)^{1-\gamma_w} = \Omega_{t+1}\Omega_{t+2}\dots\Omega_{t+j}$$

the growth factor of the wage of a a union that gets negative signals during the the next j periods (for $j=0$, we have $\Omega_{t,t}=1$, for $j=1$, we have $\Omega_{t+1,t}=\Omega_{t+1}$).

Let $\widetilde{\mathscr{U}_t}$ be the value of a union that receives a positive signal at date t and $\mathscr{U}_t(W_{t-1}(\varsigma))$ the value of a union that receives the negative signal. In the latter case, the union follows simply the *ad hoc* rule (1.41), this explains why its value at date t depends upon $W_{t-1}(\varsigma)$. On the opposite, the optimization program of a union that receives a positive signal is purely turn towards the future. As unions have the same expectations about the future, they all choose the same optimal wage ($W_t^\star$). More formally, the value at date t of a union that receives a positive signal is

$$\widetilde{\mathcal{U}}_t = \max_{\mathbf{W}} \left\{ \mathscr{S}_t\Big(\mathbf{W}\Big) + \beta \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \Big((1-\xi_w)\widetilde{\mathcal{U}}_{t+1} + \xi_w \mathcal{U}_{t+1}\big(\mathbf{W}\big) \Big) \right] \right\} \quad (1.42)$$

where Λ_t is the Lagrange multiplier associated to the nominal budget constraint of the representative household.
The value of a union that receives a negative signal is

$$\begin{aligned} \mathcal{U}_t\big(W_{t-1}(\varsigma)\big) =& \mathscr{S}_t\Big(\Gamma_t W_{t-1}(\varsigma)\Big) \\ &+ \beta \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \Big((1-\xi_w)\widetilde{\mathcal{U}}_{t+1} + \xi_w \mathcal{U}_{t+1}\big(\Gamma_t W_{t-1}(\varsigma)\big) \Big) \right] \end{aligned} \quad (1.43)$$

The first order condition and the application of the envelop theorem give

$$\mathscr{S}'_t\big(W^\star_t\big) + \beta \xi_w \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \mathcal{U}'_{t+1}\big(W^\star_t\big) \right] = 0 \quad (1.44a)$$

$$\frac{\mathcal{U}'_t\big(W_{t-1}(\varsigma)\big)}{\Gamma_t} = \mathscr{S}'_t\big(\Gamma_t W_{t-1}(\varsigma)\big) + \beta \xi_p \mathbb{E}_t \left[\frac{\Lambda_{t+1}}{\Lambda_t} \mathcal{U}'_{t+1}\big(\Gamma_t W_{t-1}(\varsigma)\big) \right] \quad (1.44b)$$

with the derivative of the union profit at $\mathcal{W}$:

$$\begin{aligned} \mathscr{S}'_t(\mathcal{W}) \quad = \quad & \varepsilon_{l,t} \frac{1-\theta_s(1+\psi_s)}{1+\psi_s} \left(\frac{\mathcal{W}}{W_t}\right)^{-(1+\psi_s)\theta_s} \Upsilon_t^{(1+\psi_s)\theta_s} \mathcal{L}_t \\ & + \theta_s \left(\frac{\mathcal{W}}{W_t}\right)^{-(1+\psi_s)\theta_s - 1} \Upsilon_t^{(1+\psi_s)\theta_s} \frac{W^m_t}{W_t} \mathcal{L}_t + \frac{\psi_s}{1+\psi_s} \varepsilon_{l,t} \mathcal{L}_t \end{aligned} \quad (1.45)$$

Let's write temporarily, in order to simplify notations, $\mathcal{W}$, the price inherited from the past. One can rewrite, one period ahead

$$\mathcal{U}'_{t+1}\big(\mathcal{W}\big) = \Omega_{t+1,t} \mathscr{S}'_{t+1}\big(\Omega_{t+1,t}\mathcal{W}\big) + \beta \xi_p \Omega_{t+1,t} \mathbb{E}_{t+1} \left[\frac{\Lambda_{t+2}}{\Lambda_{t+1}} \mathcal{U}'_{t+2}\big(\Omega_{t+1,t}\mathcal{W}\big) \right]$$

iterating toward the future and applying conditional expectation gives

$$\mathbb{E}_t \left[\mathcal{U}'_{t+1}\big(\mathcal{W}\big) \right] = \mathbb{E}_t \left[\sum_{j=0}^{\infty} (\beta \xi_p)^j \Omega_{t+1+j,t} \frac{\Lambda_{t+1+j}}{\Lambda_{t+1}} \mathscr{S}'_{t+1+j}\big(\Omega_{t+1+j,t}\mathcal{W}\big) \right]$$

Substituting in the first order condition (pour $\mathcal{W} = W^\star_t$), one gets the following condition for an optimal wage choice by a union that receives a positive signal:

$$\mathbb{E}_t \left[\sum_{j=0}^{\infty} (\beta \xi_p)^j \Omega_{t+j,t} \frac{\Lambda_{t+j}}{\Lambda_t} \mathscr{S}'_{t+j}\big(\Omega_{t+j,t} W^\star_t\big) \right] = 0 \quad (1.46)$$

The optimal wage $W_t^\star$ is the nominal wage that insures that the sum of current and expected discounted marginal profits are zero when the union can only revise nominal wages by using the *ad hoc* rule (1.41).

It is possible to obtain a recursive expression for multiplier Υ_t that appears in the expression for a union profit. Equation (1.38) can be written equivalently in the form

$$\Upsilon_t^{1-\theta_s(1+\psi_s)} = \int_0^1 \left(\frac{W_t(\varsigma)}{W_t}\right)^{1-\theta_s(1+\psi_s)} \mathrm{d}\varsigma$$

The wage offered by the union at date t appears under the integral sign. This price has been determined optimally j periods before with probability $(1-\xi_w)\xi_w^j$. We can then rewrite the integral as

$$\Upsilon_t^{1-\theta_s(1+\psi_s)} = (1-\xi_w)\sum_{j=0}^{\infty}\xi_w^j \left(\frac{\Omega_{t,t-j}W_{t-j}^\star}{W_t}\right)^{1-\theta_s(1+\psi_s)}$$

where $W_{t-j}^\star$ is the optimal wage at date $t-j$. Finally, one can interpret the infinite sum as the solution of the following recursive equation:

$$\begin{aligned}\Upsilon_t^{1-\theta_s(1+\psi_s)} =&(1-\xi_w)\left(\frac{W_t^\star}{W_t}\right)^{1-\theta_s(1+\psi_s)} \\ &+\xi_w\left(\frac{\Omega_{t,t-1}}{W_t/W_{t-1}}\right)^{1-\theta_s(1+\psi_s)}\Upsilon_{t-1}^{1-\theta_s(1+\psi_s)}\end{aligned} \tag{1.47}$$

One can get a more explicit expression for the wage that satisfies equation (1.46). Substituting in this equation the expression for marginal profit (1.45) and dividing by $W_t^{\star\,-(1+\psi_s)\theta_s}$, one gets

$$\frac{w_t^\star}{w_t} = \frac{\theta_s(1+\psi_s)}{\theta_s(1+\psi_s)-1}\frac{\mathscr{H}_{1,t}}{\mathscr{H}_{2,t}} + \frac{\psi_s}{\theta_s(1+\psi_s)-1}\left(\frac{w_t^\star}{w_t}\right)^{1+(1+\psi_s)\theta_s}\frac{\mathscr{H}_{3,t}}{\mathscr{H}_{2,t}} \tag{1.48}$$

where $w_t^\star$ is the real wage obtained by the union at date t when it can adjust the nominal wage in an optimal manner and w_t the real nominal wage in the economy, with

$$\mathscr{H}_{1,t} = \mathbb{E}_t\sum_{j=0}^{\infty}(\beta\xi_w)^j\lambda_{t+j}w_{t+j}^m\left(\frac{\Omega_{t+j}}{\frac{w_{t+j}}{w_t}\frac{P_{t+j}}{P_t}}\right)^{-(1+\psi_s)\theta_s}\Upsilon_{t+j}^{(1+\psi_s)\theta_s}\mathcal{L}_{t+j} \tag{1.49a}$$

$$\mathscr{H}_{2,t} = \mathbb{E}_t \sum_{j=0}^{\infty} (\beta\xi_w)^j \lambda_{t+j}\varepsilon_{l,t+j} w_{t+j} \times \left(\frac{\Omega_{t+j}}{\frac{w_{t+j}}{w_t}\frac{P_{t+j}}{P_t}}\right)^{1-(1+\psi_s)\theta_s} \Upsilon_{t+j}^{(1+\psi_s)\theta_s} \mathcal{L}_{t+j} \tag{1.49b}$$

$$\mathscr{H}_{3,t} = \mathbb{E}_t \sum_{j=0}^{\infty} (\beta\xi_w)^j \lambda_{t+j}\varepsilon_{l,t+j} w_{t+j} \frac{\Omega_{t+j}}{\frac{w_{t+j}}{w_t}\frac{P_{t+j}}{P_t}} \mathcal{L}_{t+j} \tag{1.49c}$$

Noticing that w_{t+j}/w_t, the growth factor of the real wage between t and $t+j$, can be equivalently written as $\Pi_{i=1}^{j} \varpi_{t+i}$ (ϖ_t is the growth factor of the real wage between t and $t-1$) and that we have

$$\Omega_{t+j,t} = (1+g)^j \left(\prod_{h=1}^{j} \mathscr{E}_{t+h}\right)^{\frac{1}{1-\rho_x}} \left(\prod_{h=0}^{j-1} \bar{\pi}_{t+h}\right)^{\gamma_w} \left(\prod_{h=0}^{j-1} \pi_{t+h}\right)^{1-\gamma_w},$$

we can finally represent variables $\mathscr{H}_{1,t}$, $\mathscr{H}_{2,t}$ and $\mathscr{H}_{3,t}$ in the recursive form

$$\mathscr{H}_{1,t} = \lambda_t w_t^m \mathcal{L}_t \Upsilon_t^{(1+\psi_s)\theta_s} + \beta\xi_w \mathbb{E}_t \left[\left(\frac{\varpi_{t+1}\pi_{t+1}}{(1+g)\mathscr{E}_{t+1}^{\frac{1}{1-\rho_x}} \pi_{t-1}^{\gamma_w} \bar{\pi}_t^{1-\gamma_w}}\right)^{(1+\psi_s)\theta_s} \mathscr{H}_{1,t+1}\right] \tag{1.50a}$$

$$\mathscr{H}_{2,t} = \lambda_t \varepsilon_{l,t} w_t \mathcal{L}_t \Upsilon_t^{(1+\psi_s)\theta_s} + \beta\xi_w \mathbb{E}_t \left[\left(\frac{\varpi_{t+1}\pi_{t+1}}{(1+g)\mathscr{E}_{t+1}^{\frac{1}{1-\rho_x}} \pi_{t-1}^{\gamma_w} \bar{\pi}_t^{1-\gamma_w}}\right)^{(1+\psi_s)\theta_s-1} \mathscr{H}_{2,t+1}\right] \tag{1.50b}$$

$$\mathscr{H}_{3,t} = \lambda_t \varepsilon_{l,t} w_t \mathcal{L}_t + \beta\xi_w \mathbb{E}_t \left[\left(\frac{(1+g)\mathscr{E}_{t+1}^{\frac{1}{1-\rho_x}} \pi_{t-1}^{\gamma_w} \bar{\pi}_t^{1-\gamma_w}}{\varpi_{t+1}\pi_{t+1}}\right) \mathscr{H}_{3,t+1}\right] \tag{1.50c}$$

Noticing that $\vartheta_{s,t} \equiv \int_0^1 \frac{W_t(\varsigma)}{W_t} \mathrm{d}\varsigma$ can be written in the recursive form

$$\vartheta_{s,t} = (1-\xi_w)\frac{w_t^\star}{w_t} + \xi_w \frac{(1+g)\mathscr{E}_t^{\frac{1}{1-\rho_x}} \pi_{t-1}^{\gamma_w} \bar{\pi}_t^{1-\gamma_w}}{\varpi_t \pi_t} \vartheta_{s,t-1} \tag{1.51}$$

we can rewrite equation (1.39) as

$$\frac{\psi_s \vartheta_{s,t}}{1+\psi_s} + \frac{\Upsilon_t}{1+\psi_s} = 1 \tag{1.52}$$

In the end, wage dynamics are characterized by equations (1.44), (1.51), (1.47), (1.48), (1.50a), (1.50b), (1.50c).

1.4 Government and monetary authority

1.4.1 Fiscal policy

We assume that exogenous government expenditures $G_t = g_t Y_t$ are exactly financed by lump sum taxes:

$$T_t = P_t G_t.$$

1.4.2 Central Bank

We assume that the behavior of the central bank is adequately described by the following Taylor rule:

$$R_t = \max\left\{1, R_{t-1}^{\rho_R}\left[R^\star \left(\frac{\pi_{t-1}}{\bar{\pi}_t}\right)^{r_\pi}\left(\frac{Y_t}{\mathscr{Y}_t}\right)^{r_Y}\right]^{1-\rho_R} \varepsilon_{R,t}\right\} \tag{1.53}$$

where $\bar{\pi}_t$ is the inflation target of the central bank, $\mathscr{Y}_t$ is the reference output level that would be attained by an economy without nominal rigidities, $\log \varepsilon_{R,t}$ is an AR(1) stationary process with zero mean. The max function constrains the nominal interest factor to be greater or equal than one. Equation (1.53) defines two regimes. In the first one, the nominal interest rate is constant (equal to zero), whereas in the second one the nominal interest rate reacts to excess inflation and output gap fluctuations. In the sequel, we consider the case where the steady state of the economy is in the second regime, *ie* the long run level of the nominal interest rate is assumed to be strictly positive[2]. Nevertheless, during the transitions to this steady state the economy can hit the Zero Lower Bound for the nominal interest rate. Because of this constraint, the model is non differentiable everywhere.

[2]Note that we would not be able to solve and simulate this model if the steady state is in the first regime. In this configuration, Blanchard and Kahn conditions would be (locally) violated.

1.5 General equilibrium

1.5.1 Price distortion

Prices in the intermediary good sector are heterogeneous. However, it is possible to show that this heterogeneity doesn't hinder aggregation. We know that the firms of this sector choose all the same mix of factor of production in the sense that the ratio of capital demand to labor demand is constant across firms (see equation 1.23). Expressing labor demand of firm ι as a function of its demand of physical capital, we can write the production of this firm

$$y_t(\iota) = \left(\frac{A_t L_t^d}{K_t^d}\right)^{1-\alpha} K_t^d(\iota)$$

When $K_t^d \equiv \int_0^1 K_t^d(\iota)\mathrm{d}\iota$ is aggregate demand for physical capital and $y_t \equiv \int_0^1 Y_t(\iota)\mathrm{d}\iota$ represents the sum of intermediary productions, we can write directly

$$y_t = \left(K_t^d\right)^{\alpha} \left(A_t L_t^d\right)^{1-\alpha}$$

The sum of intermediary productions is different from Y_t, because aggregation technology isn't linear. Integrating the demand function for good ι from the final good producers (1.18) over ι, we get

$$y_t = \Delta_{p,t} Y_t \tag{1.54}$$

with

$$\Delta_{p,t} \equiv \frac{1}{1+\psi_f}\int_0^1 \left(\left(\frac{P_t(\iota)/P_t}{\Theta_t}\right)^{-(1+\psi_f)\theta_f} + \psi_f\right)\mathrm{d}\iota \tag{1.55}$$

Price distortion can be written recursively in the following manner:

$$\Delta_{p,t} = \frac{1}{1+\psi_f}\Theta_t^{(1+\psi_f)\theta_f}\nabla_{p,t} + \frac{\psi_f}{1+\psi_f} \tag{1.56a}$$

$$\nabla_{p,t} = (1-\xi_p)\left(\frac{P_t^\star}{P_t}\right)^{-(1+\psi_f)\theta_f} + \xi_p\left(\frac{\bar{\pi}_t^{\gamma_p}\pi_{t-1}^{1-\gamma_p}}{\pi_t}\right)^{-(1+\psi_f)\theta_f}\nabla_{p,t-1} \tag{1.56b}$$

where the Lagrange multiplier Θ_t is defined recursively as well. Then, we have

$$\Delta_{p,t} Y_t = \left(K_t^d\right)^{\alpha} \left(A_t L_t^d\right)^{1-\alpha} \tag{1.57}$$

1.5.2 Wage distortion

Here, we show how to link aggregate labor supply by the households with aggregated labor supply by the employment agency to the firms of the intermediary good sector This link is affected by the heterogeneity of wages induced by their nominal rigidity. Integrating labor demand for type ς by the employment agency (1.37) over ς, we find directly

$$L_t = \Delta_{w,t} \mathcal{L}_t \tag{1.58}$$

with

$$\Delta_{w,t} \equiv \frac{1}{1+\psi_s} \int_0^1 \left(\left(\frac{W_t(\varsigma)/W_t}{\Upsilon_t} \right)^{-(1+\psi_s)\theta_s} + \psi_s \right) \mathrm{d}\varsigma \tag{1.59}$$

Wage distortion can be written in recursive form:

$$\Delta_{w,t} = \frac{1}{1+\psi_s} \Upsilon_t^{(1+\psi_s)\theta_s} \nabla_{w,t} + \frac{\psi_s}{1+\psi_s} \tag{1.60a}$$

$$\begin{aligned} \nabla_{w,t} =& (1-\xi_w) \left(\frac{w_t^\star}{w_t} \right)^{-(1+\psi_s)\theta_s} \\ &+ \xi_s \left(\frac{(1+g)\mathscr{E}_t^{\frac{1}{1-\rho_x}} \bar{\pi}_t^{\gamma_w} \pi_{t-1}^{1-\gamma_w}}{\varpi_t \pi_t} \right)^{-(1+\psi_s)\theta_s} \nabla_{w,t-1} \end{aligned} \tag{1.60b}$$

where the Lagrange multiplier Υ_t is defined recursively in equation (1.47).

1.5.3 Dividends paid by intermediary good firms

Firms in the intermediary good sector interact in monopolistic competition and make profits that are paid to households in the form of dividends. The sum of nominal profits at date t is

$$\begin{aligned} \Pi_t &= \int_0^1 \Pi_t(\iota) \mathrm{d}\iota \\ &= P_t \left(\varepsilon_{y,t} Y_t - r_t^k K_t^d - w_t L_t^d \right) \end{aligned}$$

As households own the firms, the profits are paid to them. The repartition of these profits between the households in undetermined in general equilibrium, but we know that

$$\int_0^1 \mathscr{D}_{1,t}(h) \mathrm{d}h = P_t \left(\varepsilon_{y,t} Y_t - r_t^k K_t^d - w_t L_t^d \right) \tag{1.61}$$

1.5.4 Dividends paid by the unions

In the same way, we can compute aggregate nominal profit of the unions at date t. These profits as well are paid to the households. We have

$$\begin{aligned}\mathscr{S}_t &= \int_0^1 \mathscr{S}_t(\varsigma)\mathrm{d}\varsigma \\ &= \varepsilon_{l,t} W_t \mathcal{L}_t - W_t^m L_t\end{aligned}$$

and, then,

$$\int_0^1 \mathscr{D}_{2,t}(h)\mathrm{d}h = \varepsilon_{l,t} W_t \mathcal{L}_t - W_t^m L_t \tag{1.62}$$

1.5.5 Equilibrium in factor markets and in bond markets

In general equilibrium, labor supply form the employment agency equals aggregate labor demand by firms of the intermediary good market. In the same way, aggregate supply of physical capital by the households equals aggregate demand from these firms. In formal terms,

$$\mathcal{L}_t \equiv \Delta_{w,t}^{-1} \int_0^1 L_t(h)\mathrm{d}h = \int_0^1 L_t^d(\iota)\mathrm{d}\iota \equiv L_t^d \tag{1.63}$$

$$\widetilde{K}_t \equiv \int_0^1 z_t(h) K_{t-1}(h)\mathrm{d}h = \int_0^1 K_t^d(\iota)\mathrm{d}\iota \equiv K_t^d \tag{1.64}$$

Finally, aggregate demand for bonds must be zero, as we assume a close economy and no government debt.

$$\int_0^1 B_t(h)\mathrm{d}h = 0 \tag{1.65}$$

1.5.6 Equilibrium on the good market

By summing the budget constraints of the households (1.2) over $h \in [0,1]$ and by substituting the equilibrium conditions on the bond market, the definition of aggregate dividends and the budget constraint of the government, we get

$$\begin{aligned}P_t G_t + P_t C_t + p_{I,t} P_t I_t = W_t^m L_t + P_t r_t^K z_t K_{t-1} + P_t \left(\varepsilon_{y,t} Y_t - r_t^k K_t^d - w_t L_t^d\right) \\ + \varepsilon_{lt} W_t \mathcal{L}_t - W_t^m L_t\end{aligned}$$

After simplification and knowing that the factor markets are in equilibrium, we obtain

$$G_t + C_t + p_{I,t}I_t = \varepsilon_{y,t}Y_t + (\varepsilon_{l,t} - 1)\, w_t\mathcal{L}_t \tag{1.66a}$$

or equivalently

$$G_t + C_t + p_{I,t}I_t = \varepsilon_{y,t}\Delta_{p,t}^{-1}y_t + (\varepsilon_{l,t} - 1)\, \Delta_{w,t}^{-1}w_tL_t \tag{1.66b}$$

2 Extended path

Due to the max function appearing in the Taylor rule, the model is not differentiable everywhere. Consequently, a perturbation approach cannot be considered for solving the model. The reason is that, even if we forget that the Taylor approximation does not apply in this case, by approximating the model around the deterministic steady state the ZLB (or more generally any occasionally binding constraint) would not affect the expectations of the agents.

Our model can generically be represented as

$$\mathbb{E}_t\left[\mathscr{F}(y_{t+1}, y_t, y_{t-1}, \varepsilon_t)\right] = 0 \qquad \forall t \tag{2.1}$$

where y is a vector of endogenous variables, ε is a vector of structural innovations (a Gaussian multivariate white noise), $\mathbb{E}_t$ is the conditional expectation operator and $\mathscr{F}$ is a non linear function. The standard (numerical or analytical) solution approach is to look for an expression for the decision and transition rules:

$$y_t = \mathscr{G}(y_{t-1}, \varepsilon_t)$$

where the invariant reduced form $\mathscr{G}$ satisfies:

$$\mathbb{E}_t\left[\mathscr{F}(\mathscr{G}(\mathscr{G}(y_{t-1}, \varepsilon_t), \varepsilon_{t+1}), \mathscr{G}(y_{t-1}, \varepsilon_t), y_{t-1}, \varepsilon_t)\right] = 0 \qquad \forall t$$

If there is no closed form solution, the function $\mathscr{G}$ must be locally (around the deterministic steady state) or globally (over an arbitrary domain for the states variables) approximated. Unfortunately, because of the size of our model a global approximation cannot be considered here.

The extended path approach indirectly characterizes the decision and transition rules $\mathscr{G}$ by generating time-series that satisfy the model equations.

Basically, the trick is to move the conditional expectation under function $\mathscr{F}$, that is to replace (2.1) by

$$\mathscr{F}\left(\mathbb{E}_t\left[y_{t+1}\right], y_t, y_{t-1}, \varepsilon_t\right) = 0 \qquad \forall t \tag{2.2}$$

This is obviously not the same model if the function $\mathscr{F}$ is nonlinear, because of the Jensen inequality. The main approximation of the EP method lies here. The magnitude of the errors induced by this approximation will depend on the degree of nonlinearity in the forward terms appearing in the original model. Gagnon (1990) and Love (2009), considering a stochastic growth model, show that the approximation error is reasonable and that the EP approach performs as well (or even better) as a global approximation approach (Galerkin). The advantage of the EP approach over global approximation being that it can solve medium or large scaled models (which is not possible with global approximation methods due to the so called *curse of dimensionality*). Technically, the EP method will generate time series for the endogenous variables by calling recursively a perfect foresight model solver (for (2.2)). The algorithm is described in the sequel.

Given initial conditions for the states (y_0), terminal conditions for the jumping variables ($y_{N+1} = y^\star$) and a sequence of *expected* innovations, one can solve a perfect foresight model by solving the following system of non linear equations:

$$\begin{aligned}
\mathscr{F}(y_2, y_1, y_0, \varepsilon_1) &= 0\\
\mathscr{F}(y_3, y_2, y_1, \varepsilon_2) &= 0\\
\mathscr{F}(y_4, y_3, y_1, \varepsilon_3) &= 0\\
&\vdots\\
\mathscr{F}(y_{N+1}, y_N, y_{N-1}, \varepsilon_N) &= 0
\end{aligned}$$

for the equilibrium path $\{y_1, y_2, \dots, y_N\}$. This system of nonlinear equations is solved with a Newton algorithm, taking care of the sparsity of the Jacobian. The sole approximation here concerns the terminal condition: we assume that the endogenous variables reach the steady state at time $N+1$, while this result is only asymptotic. This size of this approximation can be tailored by evaluating the sensitivity of the solution with respect to the value of the horizon, N. Alternatively, if the steady state of the model is unknown, we could replace the terminal condition by $y_{N+1} = y_N$. In practice, we may have to set N equal to several hundreds, in order to obtain a the reasonable size for the approximation errors. This explains the computational burden of this approach (the size of the non linear system of equations to solve grows linearly with N).

We now restrict our attention to the following dynamic problem (at time t):

$$\mathscr{F}(y_{t+j+1}, y_{t+j}, y_{t+j-1}, \varepsilon_{t+j}) = 0 \qquad \forall j = 0, 1, \dots, N$$

with no shocks in the future, $\varepsilon_j = 0$ for all $j > 0$. The solution of the corresponding system of non linear equations:

$$\begin{aligned}
\mathscr{F}(y_{t+1}, y_t, y_{t-1}, \varepsilon_t) &= 0 \\
\mathscr{F}(y_{t+2}, y_{t+1}, y_t, 0) &= 0 \\
&\vdots \\
\mathscr{F}(y_{t+N+1}, y_{t+N}, y_{t+N-1}, 0) &= 0
\end{aligned}$$

will be denoted:

$$\mathcal{Y}_{t,N} \equiv \{y_t, \dots, y_{t+N}\} \equiv \mathscr{Q}_{N,t}(y_{t-1}, \varepsilon_t)$$

Finally, we further restrict our attention by selecting only the first term of the solution for the equilibrium path :

$$y_t = \mathscr{R}_{N,t}(y_{t-1}, \varepsilon_t)$$

Note that the mapping between the endogenous variables at time t with the initial condition (endogenous variables at time $t-1$) and the time t innovations (ε_t) is *not state invariant*. If the initial states or the innovations are changed, we have to solve again the perfect foresight model to obtain y_t.

To simulate a time series of T observations we iterate over $\mathscr{R}_{N,t}(y_{t-1}, \varepsilon_t)$. Given an initial conditions for the states and a sequence of unexpected innovations, we have:

$$\begin{aligned}
y_1 &= \mathscr{H}_{N,1}(y_0, \varepsilon_1) \\
y_2 &= \mathscr{H}_{N,2}(y_1, \varepsilon_2) \\
&\vdots \\
y_T &= \mathscr{H}_{N,T}(y_{T-1}, \varepsilon_T)
\end{aligned}$$

The simulation of a time series with T observations requires the solutions to T perfect foresight models. Obviously, this simulation approach is far more time consuming than the perturbation (first or second order approximation) approach or the global approach (for small models). In order to reduce the computational burden of this solution method we did not use the traditional terminal condition. Knowing that the approximation error associated to the paths generated by the first order approximation of $\mathscr{G}$ are arbitrarily small in the vicinity of the steady state, we replaced the terminal condition $y_{N+1} = y^\star$ by $y_{N+1} = y_{N+1}^{(1)}$, where $y_{N+1}^{(1)}$ is the attained level of the endogenous variables at time $N+1$ when we iterate on the first order approximation of $\mathscr{G}$. Using this alternative terminal condition allows us to significantly reduce the value of N. In the same spirit, we also considered the terminal condition $y_{N+1} = \frac{\partial \mathscr{G}}{\partial y_N'}(y^\star)y_N$, but our experiments showed that this strategy was generally slower, due to the propagation of the round off errors.

3 Simulated Method of Moments

3.1 Intuition and notations

The basic idea is to find the parameters of the model such that the distance between simulated moments (mean, variance, covariance, auto-covariance, skewness, kurtosis, ...) and sample moments is minimized. It can be proved that this estimator provides consistent estimates of the parameters (see chapter 2 in Gourieroux and Monfort (1996)).

Let $\mathscr{Y}_T^\star \equiv \{y_1^\star, y_2^\star, \dots, y_T^\star\}$ be the sample and $\mathscr{Y}_T^{(s)}(\boldsymbol{\theta}) \equiv \{y_1^{(s)}(\boldsymbol{\theta}), y_2^{(s)}(\boldsymbol{\theta}), \dots, y_T^{(s)}(\boldsymbol{\theta})\}$ be a simulated sample, obtained with the extended path method described in the previous section, for a vector of (estimated) parameters $\boldsymbol{\theta}$.

Let $\mathscr{H}(\mathcal{Y}_T^\star)$ and $\mathscr{H}(\mathcal{Y}_T^{(s)}(\boldsymbol{\theta}))$ be $n \times 1$ vectors of sample and simulated moments, with $n > m$. Some of the matched moments may require more than one observation to be computed. For instance, if we want to match an order p auto-covariance function, we need to consider p consecutive values of the observed variables. To this end, we define vectors gathering p consecutive sample or simulated variables, $z_t^\star \equiv (y_t^\star, y_{t-1}^\star, \dots, y_{t-p+1}^\star)$ and $z_t^{(s)}(\boldsymbol{\theta}) \equiv (y_t^{(s)}(\boldsymbol{\theta}), y_{t-1}^{(s)}(\boldsymbol{\theta}), \dots, y_{t-p+1}^{(s)}(\boldsymbol{\theta}))$. The sample moments are defined as follows[3]

$$\mathscr{H}(\mathcal{Y}_T^\star) = T^{-1}\sum_{t=1}^{T} h(z_t)$$

[3]Assuming y_0, y_{-1}, ..., y_{-p+1} exist.

the same definition applies for the simulated moments. The continuous function $h : \mathbb{R}^{np} \longrightarrow \mathbb{R}^m$ defines the set of moments to be matched, $\mathbb{E}[h(z)]$. For instance, in the univariate case ($m = n = 1$), if the we want to match the first order auto-covariance we have $h(z_t) = (y_t - \bar{y})(y_{t-1} - \bar{y})$ where $\bar{y}$ is the arithmetic mean of y.

Finally we define the $n \times 1$ moment vector

$$g_T(\boldsymbol{\theta}) = \frac{1}{T}\sum_{t=1}^{T} h(z_t^\star) - \frac{1}{ST}\sum_{s=1}^{S}\sum_{t=1}^{T} h(z_t^{(s)}(\boldsymbol{\theta}))$$

where S is the number of simulated samples. Note that the seed of the random number routines used to generate simulated samples must be kept constant when the moment vector is evaluated for different values of the vector of parameters $\boldsymbol{\theta}$. We will denote $G(\boldsymbol{\theta})$ the $n \times m$ Jacobian matrix associated to the moment vector:

$$G(\boldsymbol{\theta}) = \frac{\partial g}{\partial \theta'}(\boldsymbol{\theta})$$

3.2 The simulated moments estimator

The simulated moments estimator of $\boldsymbol{\theta}$ is defined as:

$$\hat{\boldsymbol{\theta}}_T(W) = \arg\min_{\boldsymbol{\theta}} \, g_T(\boldsymbol{\theta})' W g_T(\boldsymbol{\theta}) \tag{3.1}$$

where W is a symmetric positive definite weighting matrix. It can be shown that this estimator is $\mathcal{O}_p(T^{-1/2})$ and that its asymptotic variance is given by:

$$\mathbb{V}_\infty\left[\sqrt{T}\ \hat{\boldsymbol{\theta}}_T(W)\right] = \kappa(S)\left(G_0'WG_0\right)^{-1} G_0'W\Omega W G_0 \left(G_0'WG_0\right)^{-1} \tag{3.2}$$

where

$$\kappa(S) = \left(1 + \frac{1}{S}\right)$$

is a scale factor monotonically decreasing with the number of simulated samples used to evaluate the moment vector,

$$G_0 = G(\boldsymbol{\theta}_0)$$

is the Jacobian matrix of the moment vector evaluated at the true value $\boldsymbol{\theta}_0$, and Ω is the long run covariance matrix of the vector of moments:

$$\Omega = \lim_{T\to\infty} \mathbb{E}\left[\left(T^{-\frac{1}{2}}\sum_{t=1}^{T} h(z_t)\right)\left(T^{-\frac{1}{2}}\sum_{t=1}^{T} h(z_t)\right)'\right]$$

3.3 Weighting matrix

Conditionally on a weighting matrix W, the variance of $\hat{\boldsymbol{\theta}}_T$ is given by (3.2). It can be shown that this variance is minimized when the weighting matrix is the inverse of the long run variance of the sample moments. This long run covariance matrix is estimated as follows

$$\widehat{\Omega} = \widehat{\Gamma}_0 + \sum_{j=1}^{p} \left(1 - \frac{j}{p+1}\right) \left(\widehat{\Gamma}_j + \widehat{\Gamma}_j'\right)$$

where the bandwidth parameter p is $\mathcal{O}(T^{1/4})$ and the lag j estimated auto-covariance is given by:

$$\widehat{\Gamma}_j = \frac{1}{T} \sum_{t=j+1}^{T} \left(\mathscr{H}(z_t) - \overline{\mathscr{H}(z)}\right) \left(\mathscr{H}(z_t) - \overline{\mathscr{H}(z)}\right)'$$

Provided $\widehat{\Omega}$ is used as a weighting matrix, the asymptotic behavior of the SMM estimator is given by:

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0\right) \underset{T\to\infty}{\Longrightarrow} \mathcal{N}\left(0, \kappa(S) G_0' \widehat{\Omega} G_0\right)$$

where the gradient of the moment function evaluated at the true vector of parameters, G_0, may be replaced by $G(\hat{\boldsymbol{\theta}})$.

4 Estimation results

As a first attempt we consider a simplified version of the model. First, we remove all the shocks except the stationary productivity shock, the risk premium shock and the investment efficiency shock. Second we only estimate the parameters related to these three shocks (autoregressive parameters and standard deviations of the innovations). The other parameters are calibrated at the posterior mean obtained in Adjemian and Juillard (2009), see table 1.

The model is estimated with quarterly data on the period 1994–2008 (3rd quarter). We use the following observed variables: GDP, private consumption expenditures, private investment (sum of private growth capital formation in residential buildings and in plant and equipment), con- sumer price index. For the short term nominal interest rate we use the Bank of Japan target rate of unsecured overnight call rate.

The parameters are estimated using the following moment conditions: (i) Mean of all the observed variables except growth of real wages, (ii) Variance of all the observed variables, (iii) First order auto-covariances for all the observed variables, (iv) Skewness of the growth rate of output and the inflation factor and (v) Kurtosis of the growth rate of output and the inflation factor. We do not use third and fourth order moments on the nominal interest factor because we want to test the ability of this model to hit the ZLB.

The estimation for the autoregressive parameters and innovation standard deviations are as follows:

ρ_A	0.9982
ρ_B	0.4912
ρ_I	0.6289
σ_A	0.0013
σ_B	0.0017
σ_I	0.0001

The estimated values are sensibly different from the values reported in Adjemian and Juillard (2009). By simulating a long time series (10000 periods) we find that the probability of hitting the ZLB is around 24%. But the probability of staying on the ZLB five quarters is around 0.15%, and the probability of being on the ZLB for a couple of years is zero. This is clearly at odds with the data. In this model, the economy is driven on the ZLB by the risk premium shock, which (partially) accounts for financial imperfections. In order to simulate paths with long periods on the ZLB, we would need much larger (positive) risk premium shocks. Another solution would be to abandon the Gaussian assumption and allow the distribution of the risk premium shock to be asymmetric. The risk premium shock should then be more often positive than negative, reflecting the long period of financial troubles in the Japanese economy.

Our results need to be checked by (i) enlarging the set of estimated parameters, and (ii) trying other moment conditions. We also need to evaluate the robustness of our simulation strategy by comparing the EP approach to a more conventional global approximation approach (finite elements) on a smaller model admitting the same kind of non linearities.

Parameter name	Value
σ_c	0.8514
σ_l	1.7180
η	0.1836
ξ_p	0.4056
ξ_w	0.7840
ψ_f	-4.8380
ψ_s	-5.5056
θ_f	6.1786
θ_s	5.9342
α	0.3030
γ_p	0.2079
γ_w	0.2757
ρ_R	0.5807
r_π	1.4290
r_y	1.7418
g	0.0031
ψ	0.2361
$\bar{z}$	0.8012

Table 1: Calibration of the parameters.

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